Glutoses: a Generalization of Topos theory

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Abstract

A generalization of topos theory is proposed giving an abstract realization of such categories as, say, the categories of manifolds and of Grothendieck schemes on the one hand, and permitting one, on the other hand, a view on "non-commutative" or, more generally, "universal" algebraic geometry, which is alternative to already existing, and is closer, in some sense, to the classical Grothendieck's construction of commutative schemes. Another immediate application of the theory developed is construction of an extension of the category of Grothendieck schemes to the category of "étale schemes" containing together with any scheme every étale sheaf over it as well.

The main result of this work is that for any presite satisfying some smallness conditions (existence of *local* sets of topological generators) there exists the universal "completion" of a presite to a glutos.

This paper is a corrected and extended version of JINR-preprint E5-93-45 (1993) [19]. It is more or less complete as concerns formulations of results, but proofs are only hinted in several places. A version which includes proofs as well is now in progress.

1 History & Motivations

The theory presented here originates from autor's works on the theory of infinite-dimensional supermanifolds ([15]-[17]). It was purposed originally just to develop the technical tools permitting one to deal automatically with numerous kinds of "charts and atlases" arising in the theory of supermanifolds (in the latter theory the role of the category of sets is played by some functor category, equipped with some pretopology).

The scheme is as follows: given a category $\mathcal{C}$ equipped with some *pre*topology (which has *not*, generally speaking, a set of topological generators) to construct another category

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$\widetilde{\mathcal{C}}$ with a pretopology together with the universal continuous functor

$$Y_{\mathcal{C}}\colon \mathcal{C} \hookrightarrow \widetilde{\mathcal{C}} \tag{1}$$

such that, roughly speaking, $\widetilde{\mathcal{C}}$ is complete with respect to "gluing along open (= belonging to some covering) arrows" and, besides, its objects are "locally isomorphic" to objects of $\mathcal{C}$.

A bit more precisely, this means that certain functors with values in open arrows of $\widetilde{\mathcal{C}}$ (called 'glueing data' or 'gluons') have colimits satisfying certain conditions. I have called such universal completion 'glutos associated with $\mathcal{C}$'.

An archetypical example is the case when $\mathcal{C}$ is the category of open regions of Euclidean spaces (resp. of Banach spaces, resp. of Banach superspaces) with (super)smooth morphisms. Then $\widetilde{\mathcal{C}}$ is naturally equivalent to the category **Man** of smooth manifolds (resp. to the category **BMan** of smooth Banach manifolds, resp. to the category **SMan** of Banach supermanifolds), whereas any glueing data can be identified with some atlas on its colimit.

Essential point here is that the glutos $\widetilde{\mathcal{C}}$ is determined completely (up to natural equivalence, of course) by the category $\mathcal{C}$ and its pretopology.

Originally there were wery strong restrictions imposed on the pretopology of $\mathcal{C}$ in order that the construction of $\widetilde{\mathcal{C}}$ by means of "charts and atlases" routine make sense. In particular: a) the pretopology on $\mathcal{C}$ must be subcanonical; b) open arrows should be mono; c) union of open arrows must exist and be open again (as is the case in the "generic" pretopology on **Top**); d) open subobjects of any object of $\mathcal{C}$ must form a set.

Later on the construction of charts and atlaces was extended, so as to include pretopologies not satisfying condition c) above. This is, e.g., the case when $\mathcal{C}$ is the category dual to that of commutative rings, with Zariski pretopology on it. It turns out that in this case the category $\widetilde{\mathcal{C}}$ is naturally equivalent to the category of Grothendieck schemes.

The latter result forces a natural question: whether one can remove condition a) on the pretopology of $\mathcal{C}$ to construct *non-commutative* schemes by means of the "glutos generator" (1) (there are several analogues of Zariski pretopology for the category dual to the category of *all* rings (see [3]), neither of them is subcanonical), as well as to get rid of condition b), to be able to produce the category of algebraic spaces (or, may be some its extension) by means of the same universal glutos construction.

The answer to the first of this questions is positive.

It turned out that, in case of a pretopology satisfying conditions a) and d) above, the universal arrow (1) exists and can be represented as the composition of three universal arrows:

$$Y_{\mathcal{C}}\colon \mathcal{C} \xrightarrow{Y'_{\mathcal{C}}} \mathcal{C}_{sub} \xrightarrow{Y''_{\mathcal{C}}} \overline{\mathcal{C}} \xrightarrow{Y'''_{\mathcal{C}}} \widetilde{\mathcal{C}}, \tag{2}$$

where the functor $Y'_{\mathcal{C}}$ is the universal arrow into presites with subcanonical pretopology (cf. Sect. A), the functor $Y''_{\mathcal{C}}$ is the universal completion of $\mathcal{C}_{sub}$ by objects which are "unions of open subobjects", whereas the functor $Y''_{\mathcal{C}}$ is the "charts and atlases" construction in "canonical" sense.

The scetch of this early results was published in 1986 in [18]. The manuscript containing detailed results and its proofs (named "Manifolds IV") was never published.[1]

[1]I am very grateful to D.Leites, who organized translation of it into ChiWriter in 1988 (in those

During next several years I have made a number of attempts to generalize the main result of [18], trying, in particular, to get rid of the condition b) (open arrows are mono) on the pretopology of $\mathcal{C}$ while *extending*, simultaneously, the very notion of a glutos. All this attempts were non-satisfactory, because, as I understand now, I was too hypnotized by "charts and atlases" paradigm. It turned out now that, in the general case, there are much more new objects in the glutos $\widetilde{\mathcal{C}}$, then can be obtained by using atlases alone (i.e. glueing along open arrows in $\mathcal{C}$ only). The whole process of completion of $\mathcal{C}$ to $\widetilde{\mathcal{C}}$ turns out to be transfinite!

Meanwhile, in 1993 there appeared a paper of Paul Fait [7] devoted to the same problem, whose main result (Theorem 14.4 on p.94) was the proof of existence of the completion $Y_{\mathcal{C}}: \mathcal{C} \hookrightarrow \widetilde{\mathcal{C}}$ for a presite $\mathcal{C}$ with subcanonical pretopology but at formally weaker other conditions imposed on the pretopology of $\mathcal{C}$ than those in [18] (they reduce exactly to conditions in [18], plus the requirement that the pretopology of $\mathcal{C}$ be subcanonical, if one assumes that open arrows are mono).

Unfortunately, the work [7] uses highly non-standard terms even for quite standard categorical notions, which makes it difficult for reading. Besides, just to understand what the main theorem 14.4 of [7] is talking about, one needs to look through the whole paper [7] in search of definitions.

That's why below is given the translation (to a reasonable extent) of the main result of [7] to common math language purposing to compare this result with that of [18].

A category $\mathcal{C}$ equipped with a subcanonical ("intrinsic" in terms of P. Fait) pretopology τ is called by P. Fait a **local structure** if

a) for any object X of $\mathcal{C}$ equivalence classes of open arrows $u: U \longrightarrow X$ form a small set (smallness condition);

b) Any family of open arrows such that there exists a refinement of it which is a covering of τ is a covering itself (such pretopology is called "flush" by P.Fait; in the present paper the corresponding property is just added to the definition of a pretopology — see condition (PT4) in Sec. 9);

c) Every open retraction u is a covering morphism (i.e. the singleton $\{u\}$ is a covering);

d1) Every arrow $u: U \longrightarrow X$ which is a covering locally is a covering of X. Here "u is a covering locally" means that there exists a covering $\{u_i: U_i \longrightarrow X\}_{i \in I}$ of X such that for any $i \in I$ the pullback projection $U_i \prod_X U \longrightarrow U_i$ is a covering of U_i;

d2) Let $u: U \longrightarrow X$ be an arrow such that there exists a covering $\{u_i: U_i \longrightarrow U\}_{i \in I}$ such that for any $i \in I$ the restriction arrow $uu_i: U_i \longrightarrow X$ is open and, besides, the pullback projection $U_i \prod_X U \longrightarrow U_i$ is a covering of U_i. Then u is an open arrow.

(conditions d1)-d2) together is the "CLCS condition" in the terminology of P.Fait);

e) For the last condition of P. Fait: "$\mathcal{C}$ is complete under Cvm" I failed to find a simple formulation in terms of "internal" properties of $\mathcal{C}$ itself; as formulated by

times TEX was not yet, unfortunately, as popular as it is now.). I hope the corresponding ChiWriter file, translated by Chi2TeX programm (though badly: Chi2TeX knows nothing about commutative diagrams!), will reduce essentially the work needed to write the complete version of this work on glutoses.

P. Fait it includes a complicated construct, namely, some functor $+\colon \mathcal{C} \longrightarrow \mathcal{C}^+$ called by P. Fait "the plus functor". The construction of this "plus functor" takes a considerable part of the paper [7]. First, there is constructed the presite $\mathcal{C}^p$, whose objects are just glueing data on the presite $\mathcal{C}$ ("canopies" in the terminology of P. Fait) and whose arrows and pretopology are "forced" in some sense by that of $\mathcal{C}$. The pretopology on $\mathcal{C}^p$ fails to be subcanonical, so the next step is to form a new category $\mathcal{C}^+$ out of $\mathcal{C}^p$, this time with subcanonical pretopology by some process, called "smoothing" by P. Fait. I suspect (though have not checked it) that "smoothing" is equivalent to a particular case of our construction of universal presite with subcanonical pretopology (see Section A below). Very roughly and informally the "plus functor" turns out to be the "square root" of the desired functor $Y_{\mathcal{C}}\colon \mathcal{C} \hookrightarrow \widetilde{\mathcal{C}}$.

In terms of this "plus functor" the last condition in the definition of local structure says that the functor $+$ reflects covering morphisms, in the sense that a single arrow $u\colon U \longrightarrow +(X)$ of $\mathcal{C}^+$ is a covering of $+(X)$ iff it is isomorphic (in $\mathcal{C}^+/+(X)$) to an arrow $+(u')$ for some arrow $u'\colon U' \longrightarrow X$ of $\mathcal{C}$.

A local structure $\mathcal{C}$ is called by P. Fait a **global structure** if it "is complete under affinization", which means in our terms that every open glueing data in $\mathcal{C}$ (see definition in Sec. 8 below) has a universal colimit satisfying certain natural conditions (strong form of our conditions $(\mathbf{G4}_{\mathcal{U}}+\mathbf{5})$ in Sect. 8 below).

The p. (E) of the main theorem 14.4 of [7] states then that *for any local structure* $\mathcal{C}$ *the category* $\mathcal{C}^{++}$ *is a global structure, whereas the functor* $++\colon \mathcal{C} \longrightarrow \mathcal{C}^{++}$ *is universal among continuous functors to global structures*.

One can see that if one imposes on a local structure the additional requirement that all open arrows are mono, then conditions c)–e) in the definition of the local structure above can be omitted. In this particular case 'local structure' resp. 'global structure' of P. Fait is exactly the same thing as 'preglutos with subcanonical topology' resp. 'glutos' in my paper [18] (in the current paper glutoses in the sense of [18] are called 'nearly $\mathcal{U}$-glutoses', and they correspond to a particular case of glutoses in the present sense, namely, SG-glutoses).

On the other hand, without the requirement that opens are mono, the condition e) in the definition of local structure seems to me to be too *ad hoc* and, besides, I see no simple way to check whether a given presite satisfies it. For example, the category of algebraic spaces with étale pretopology on it is declared to be a global structure (Example E on p.2 of [7]), but I have found no indication in the text of [7] why the condition e) is valid for this case.

In short, it is not clear at all, to what extent conditions imposed on the pretopology of a local structure are weaker compared to the same conditions plus the condition stating that opens are mono.

In any case, the proof in [7] of existence of the universal global structure is performed completely inside a given universe, without using the axiom of existence of strongly inaccessible cardinals, whereas I use systematically in proofs "big" toposes (not belonging, generally speaking, to the universe where 'local' and 'global' structures live), which makes the results of [7] stronger than mine from purely set-theoretic point of view.

My own attempts to generalize the results of my paper [18] were based on the following experimental fact (discovered after the publication of [18]): for any object X of a 'glutos' $\mathcal{C}$ the full subcategory of the category $\mathcal{C}/X$ consisting of *locally open* arrows $f\colon Y \longrightarrow X$ turnes out to be a *topos*. This generalizes the (archetypical!) fact that the topos of sheaves over a topological space X is naturally equivalent to the category of local homeomorphisms over X.

Toposes arising in such a way are of a very special kind — they are so called SG-toposes (i.e. those toposes, for which subobjects of 1 form a set of generators). So it seems to be natural to look for such an extension of the theory, where 'locally open' arrows over any object of a 'generalized glutos' form a topos again, this time *arbitrary* Grothendieck topos.

It took me several years both to find the right generalization of glutos definition and to get rid of *all* restrictions on the pretopology of $\mathcal{C}$ in the "universal glutos generator" (1), excepting a highly weakened form of smalness condition d) above (the existence, for any object X of $\mathcal{C}$, of a small set of topological generators in the induced presite of open arrows over X).

The axioms of the theory arising admit an elementary version, in which case the theory may be viewed as some generalization of the theory of elementary toposes (see Sect. 4), whereas the version of its axioms including small (with respect to some universe $\mathcal{U}$) families of objects or arrows in its formulation (see Sect. 7) is a generalization of Grothendieck's toposes.

And it turned out that in the general case charts and atlases are non-adequate for constructing the glutos $\widetilde{\mathcal{C}}$ out of $\mathcal{C}$.

Unfortunately, the general definition of a glutos given in the first version of this paper [19] turned out to be wrong. It failed to satisfy to what physicists would call "the correspondence principle": the old theory must be a particular case of the new one, meaning here that the 2-category of "old" glutoses (i.e. in the sense of [18]) must be (naturally equivalent to) a full 2-subcategory of "new" ones (i.e. in the sense of [19]).

Namely, one of the axioms of glutoses (a part of the axiom (G5)) as formulated in [19] turned out to be too strong, so that all classical examples (topological spaces, manifolds, etc.) failed to satisfy it[2], though "glutos generator" above works for this definition, too!

It took me only about a month to correct the definition, finding a weaker theory, satisfying the correspondence principle, but the writing of this corrected version of [19] was delayed for more than 5-years due to various reasons of non-mathematical nature.

As to the "wrong" glutoses, they are not thrown away. Instead they are present in this paper under the name 'ultraglutoses'. Though they do not satisfy correspondence principle for old version of glutos theory, scetched in [18], they *do satisfy* this principle, being one of the correct extensions of the *topos theory* instead!

Glutoses do satisfy the latter principle as well, but, in some sense, the theory of ultraglutoses is a better extension of the topos theory, because they are more "ideal" from purely categorical point of view (more diagrams have colimits inside ultraglutoses

[2]The counterexample was found by Prof. P. T. Johnstone (1983, private communication); I am very grateful to him for discovering this nasty bug in my definitions, leading to bugs in a number of statements in the paper [19])

and this colimits are "good"). The most important thing is that any glutos has the universal fully faithful imbedding into an ultraglutos.

But ultraglutoses themselves seem to be not the end of the story yet.

For example, the axiom of finite completeness is not included in their definition, just because I wrongly believed originally that such categories as **Man**, etc. are to be models of this theory.

Now, there are no principal obstructions to add the finite completeness axiom to the theory of ultraglutoses, as far as one proves that *any glutos* $\mathcal{C}$ *imbeds* (fully and faithfully) *into finitely complete ultraglutos* $\mathcal{C}'$*, so that the imbedding is universal among all continuous functors* (not necessarily exact!) *into ultraglutos.* I believe the latter statement is true, but have not find time yet to check it.

2 Set-theoretic conventions

We will work here within Morse set theory (see [20]), with the usual axiom added stating the existence for any set of a universal *set* containing it.

The latter axiom seems to be not necessary for the validity of our main Theorem 10.13 (reformulated properly), but its use highly facilitates our construction of universal glutoses and ultraglutoses.

Namely, to constuct a universal (ultra)glutos $\widetilde{\mathcal{C}}$ for a presite $\mathcal{C}$ living in some universe $\mathcal{U}$, we use as building blocks objects and arrows of the topos $\mathrm{Sh}_{\mathcal{U}'}\mathcal{C}$ of shieves on $\mathcal{C}$ with values in some *higher* universe $\mathcal{U}'$, containing $\mathcal{U}$ as an element. This simplifies a number of proofs.

All categories, presites, toposes, glutoses, etc. are supposed to be sets, so that there exists the legitimate 2-category of all categories, resp. presites, etc. (which is a proper class); similarly, (pseudo)functorial operations defined in several places below on objects, arrows and 2-arrows of 2-categories above are incorporating together to produce legitimate 2-(pseudo)functors.

Here are assumed the definitions of [20] for ordered pairs and, more generally, families (= tuples in the terminology of [20]) so that for any universe $\mathcal{U}$ (including the biggest universe of *all* sets) a family of subclasses of the universe $\mathcal{U}$ indexed by a subclass of the universe $\mathcal{U}$ is again a subclass of $\mathcal{U}$ and behaves well[3]. In fact, it is just this choice of definition for families which permits one to *define* the "big" 2-categories above and 2-(pseudo)functors between them as terms of Morse set theory: e.g., a 2-category $\mathcal{C}$ *is* a finite tuple $\langle \mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \ldots \rangle$ of classes satisfying certain conditions.

3 Inverse images of equivalence relations

Here are given some definitions and elementary results, necessary to formulate the axioms of glutoses.

[3]This means that families are "separated": $\{\mathcal{C}_i\}_{i\in\mathcal{I}} = \{\mathcal{C}'_i\}_{i\in\mathcal{I}'}$ iff $\mathcal{I} = \mathcal{I}'$ and for any $i \in \mathcal{I}$ one has $\mathcal{C}_i = \mathcal{C}'_i$. In other words, there is no loss of information while encoding some data inside a family, even the big one.

Proposition 3.1 *Let $d_0, d_1 \colon U \rightrightarrows X$ be an equivalence relation in a category $\mathcal{C}$ (not necessarily with finite products). Let $f \colon X' \longrightarrow X$ be a pullbackable arrow of $\mathcal{C}$. Consider the diagram of three pullbacks*

$$
\begin{tikzcd}
 & & U' \arrow[dl] \arrow[dr] & & \\
 & U'_0 \arrow[dl] \arrow[dr] & & U'_1 \arrow[dl] \arrow[dr] & \\
X' \arrow[dr, "f"] & & U \arrow[dl, "d_0"'] \arrow[dr, "d_1"] & & X' \arrow[dl, "f"'] \\
 & X & & X &
\end{tikzcd}
\tag{3}
$$

Let $d'_0 \colon U' \longrightarrow X'$ (resp. $d'_1 \colon U' \longrightarrow X'$) *be the composition of left* (resp. *right*) *pullback projections of the diagram above. Then the pair $d'_0, d'_1 \colon U' \rightrightarrows X'$ is an equivalence relation on X' which will be called* **induced on** X' **by** (d_0, d_1) **along** f *or* **inverse image of** (d_0, d_1) **along** f *and will be denoted $f^*(d_0, d_1)$* (Note that $f^*(d_0, d_1) \neq (f^* d_0, f^* d_1)$!).

Proof. Imbed $\mathcal{C}$ into some "big" topos $\widehat{\mathcal{C}} = [\mathcal{C}^\circ, \mathbf{Set}_{\mathcal{U}}]$ of preshieves on $\mathcal{C}$ with values in a universe $\mathcal{U}$ via Yoneda (recall that due to our choice of set theory such universe exists if the category $\mathcal{C}$ is a set). Augmenting the diagram (3) by a coequalizer $r \colon X \longrightarrow R$ of (d_0, d_1) in $\widehat{\mathcal{C}}$ and using Giraud theorem as well as left exactness of Yoneda functor we conclude that (d'_0, d'_1) is a kernel pair (in $\widehat{\mathcal{C}}$) of rf which implies that (d'_0, d'_1) is an equivalence relation. ∎

4 Glutoses: definition

An **elementary glutos** is a kind of "generalized elementary topos" (not to confuse with quasitoposes!): it is a category $\mathcal{C}$, equipped with a suitable structure, given by a subset $\mathcal{O}$ of arrows of $\mathcal{C}$ (elements of which will be called **open** arrows of $\mathcal{C}$), which generates over any object X of $\mathcal{C}$ an elementary topos, namely, $\mathcal{O}/X$. Here $\mathcal{O}/X$ denotes the full subcategory of $\mathcal{C}/X$ formed by all objects which are arrows of $\mathcal{O}$. So that, in some sense, a glutos is a topos *locally* (not to be confused with local toposes!). If one grasps, metaphorically, a glutos as a family of toposes coherently glued together into a single category $\mathcal{C}$ by means of a "glueing" structure $\mathcal{O}$, then the term 'glutos' itself can be thought of as an abbreviation for 'GLUed bunch of TOposeS'. An alternative interpretation of this term: in glutoses one can *glue* together finite families of objects along open arrows (see section 8 below for details).

Exact conditions $\mathcal{O}$ must satisfy ("axioms of elementary glutoses"), are the following conditions **(G1)**-**(G5P)** below.

But we need first to give some definitions (to formulate correctly condition **(G5c)**). An equivalence relation $u, v \colon U \rightrightarrows X$ will be said to be $\mathcal{O}$-**coequalizable**, if there exists an arrow $q \colon X \longrightarrow Q$ belonging to $\mathcal{O}$ such that $qu = qv$; the relation $u, v \colon U \rightrightarrows X$ will be said to be **(finitely) locally** $\mathcal{O}$-**coequalizable** if there exists a (finite) epi family $\{u_i \colon U_i \longrightarrow X\}_{i \in I}$ of arrows of $\mathcal{O}$ such that every induced equivalence relation $u_i^*(u, v)$ is $\mathcal{O}$ coequalizable (in the latter definition it is supposed that both u and v are pullbackable so that the corresponding induced equivalence relations do exist).

(G1) $\mathcal{O}$ contains all iso's of $\mathcal{C}$, is contained in the set of all pullbackable arrows of $\mathcal{C}$ and is stable by composition and pullbacks;

(G2) $fg \in \mathcal{O}$ and $f \in \mathcal{O}$ implies $g \in \mathcal{O}$;

(G3) a) For any object X of $\mathcal{C}$ the category $\mathcal{O}/X$ is an elementary topos and

b) for any $f\colon X \longrightarrow Y$ in $\mathcal{C}$ the functor $f^*\colon \mathcal{O}/Y \longrightarrow \mathcal{O}/X$ (which is defined, due to (G1), and is left exact) is an inverse image of some geometric morphism;

(G4) a) $\mathcal{C}$ has disjoint and universal finite coproducts, such that canonical injection morphisms belong to $\mathcal{O}$;

b) for any finite family $\{U_i \xrightarrow{u_i} X\}_{i\in I}$ of arrows of $\mathcal{O}$ the colimit arrow $\coprod_i U_i \longrightarrow X$ belongs to $\mathcal{O}$.

(G5) a) any epi of $\mathcal{C}$, which belongs to $\mathcal{O}$, is effective;

b) if both fp and p belong to $\mathcal{O}$ and p is epi then f belongs to $\mathcal{O}$;

c) Any equivalence relation $u, v\colon U \rightrightarrows X$ in $\mathcal{C}$ which is **open** (i.e. $u, v \in \mathcal{O}$) and finitely locally $\mathcal{O}$-coequalizable, is effective and has a universal coequalizer in $\mathcal{C}$ which belongs to $\mathcal{O}$.

(G5P) For any pullback diagram

$$\begin{tikzcd} V \arrow[rr] \arrow[d] & & Y \arrow[d, "f"] \\ X \arrow[r, two heads, "r"] & R \arrow[r, "g"] & S \end{tikzcd} \tag{4}$$

such that r is an open epi there exists a pullback of f along g.

A glutos $(\mathcal{C}, \mathcal{O})$ will be called an **ultraglutos** if p. c) of the axiom (G5) is replaced by the stronger axiom:

cu) Any open equivalence relation $u, v\colon U \rightrightarrows X$ in $\mathcal{C}$ is effective and has a universal coequalizer in $\mathcal{C}$ which belongs to $\mathcal{O}$.

In what follows we will say, if necessary, 'weak axiom (G5)' or 'strong axiom (G5)' to distinguish between glutoses and ultraglutoses.

It is clear, that elementary (ultra)glutoses are really models of some first-order theory, which is an extension of the elementary theory of categories by some unary relation symbol $\mathcal{O}$ ($\mathcal{O}(f)$ meaning 'f is an open arrow'), with corresponding translations of (G1)–(G5P) added as axioms.

Remark 4.1 Structures $\mathcal{O}$ on a category $\mathcal{C}$ satisfying condition (G1) occur so frequently that deserve, in author's opinion, to be christened somehow. Here it is proposed to call them **cloposes**, whereas for its elements is reserved the name **clopen** arrows. An argument in favour of this strange choice of names is that in the category of topological spaces both the class of all arrows isomorphic to inclusions of closed subspaces and that isomorphic to inclusions of open subspaces satisfy condition (G1). The terms 'closed' and 'open' can then be reserved to denote something more special than simply elements of a

class of arrows satisfying condition (G1) (e.g. closed arrows of a closure operator [14] or open arrows in the sense of [12] [4] or in the (different) sense of the present work).

In the present work the term ‘open’ will often be used instead of ‘clopen’, at least in the contexts of both glutoses and presites.

Remark 4.2 One can prove that axiom (G5P) follows from another axioms in case of ultraglutoses (see Prop. 8.6 below) as well as in case of SG-glutoses defined in Sect. 11 below.

Remark 4.3 So we have two theories now, both pretending to be something like “topos theory for the 2-category of closes”. Which one of them is “the” theory?

Glutos theory seems to be fixed, because it was designed to have as its models such creatures, existing in real Universe, as **Man**, **Top**, etc. So that I see no way to make it stronger.

Ultraglutos theory has no such restrictions. Having no other experimentally discovered models so far, excepting toposes themselves, it can permit additional axioms to be added, as far as ultraglutoses, generated via “ultraglutos generator” (1) from presites (or glutoses) belonging to the Universe, belong to the Universe as well.

One evident candidate for such an extension is the axiom of finite completeness, as I have already noted in Sect. 1.

As soon as it will be proved that any ultraglutos has the universal imbedding into complete ultraglutos, the complete ultraglutoses may pretend that they are the “ideal” extension of glutos theory.

One is to stress, however, that left exactness must not be included in the definition of morphisms between complete ultraglutoses, as far as we will have the possibility to continue such morphisms of glutoses as

$$\mathrm{spec}{:}\mathbf{Schem} \longrightarrow \mathbf{Top}\ ,$$

to morphisms of finitely complete ultraglutoses.

5 Examples and Counterexamples

(0) Any topos is, canonically, an ultraglutos (set $\mathcal{O} = \mathcal{C}$); vice versa, if a pair $(\mathcal{C}, \mathcal{C})$ is a glutos, then $\mathcal{C}$ is a topos iff it has a terminal object (the latter condition is necessary as one can see from example at the end of section 6). For another examples of glutos structures on toposes (étale structures satisfying collection axiom of [12]) see Remark 7.6 in section 7 below.

I know of no other “natural” example of ultroglutoses, though, as our main theorem 10.13 shows, every glutos has a universal full imbedding into an ultraglutos.

Archetypical examples of glutoses which are not toposes are:

(1) Topological spaces (**Top**) with open arrows being local homeomorphisms;

[4] I am grateful to Prof.P.T. Johnstone who turned my attention to the preprint [12] sending me a copy of it.

(2) Smooth manifolds (**Man**) with open arrows being local diffeomorphisms; for a natural number n the full subcategory $\mathbf{Man}_n$ of **Man** consisting of all manifolds of dimension n with the empty manifold $\emptyset$ added, with open arrows as above. The glutos $\mathbf{Man}_0$ degenerates, evidently, to the topos **Set** of sets, whereas $\mathbf{Man}_n$ for $n \neq 0$ give examples of glutoses without terminal objects;
(3) Locally trivial vector bundles over smooth manifolds (**Vbun**) with open arrows being just those arrows, whose image in **Man** under forgetful functor is open;
(4) Grothendieck schemes (**Schem**) or C^∞-schemes of Dubuc [6] (C^∞-**Schem**) with, e.g., open arrows in **Schem** being morphisms which locally are inclusions of open subschemes: i.e., $(u: U \longrightarrow X) \in \mathcal{O}$, if there exists a covering of U by open subschemes such that the restriction of u on any element of this covering is isomorphic to inclusion of an open subscheme of X.

It is, implicitly, assumed in examples (1)–(4) above that, say, all topological spaces of **Top** are elements of some universe $\mathcal{U}$, which, moreover, contains, for the cases (2)–(3) as well as C^∞-**Schem**, the universe $\mathcal{U}_f$ of finite sets as an element. So that we will write further, if necessary, $\mathbf{Top}_{\mathcal{U}}$, resp. $\mathbf{Set}_{\mathcal{U}}$, etc., instead of **Top**, resp. **Set**, etc..

Remark 5.4 (P. T. Johnstone's counterexample) The fact that categories **Man** and **Top** does not satisfy the strong form uc) of axiom (G5), i.e. are not ultraglutoses in current terminology, was told to me by P. T. Johnstone in 1993.

The following proposition is a generalization of this counterexample.

Proposition 5.2 *Let G be a discrete Lie group acting smoothly on a manifold M via $\mu: G \times M \longrightarrow M$. Then:*

a) *both μ and projection map $\pi_M: G \times M \longrightarrow M$ are local diffeomorphisms;*

b) *If G acts freely on M, then the pair (π_M, μ) is an equivalence relation on M, whose coequalizer in* **Top** *is the space of orbits of G equipped with the factor topology.*

Of course, any smooth action of any Lie group G on a manifold lifts to the smooth action of the same group G equipped with the discrete topology.

P. T. Johnstone's counterexample is obtained, if one takes $M = S^1$ (circle with unit length), $G = \mathbb{Z}$, and the action of $\mathbb{Z}$ on S^1 is generated by shift (=rotation) on irrational distance. It is clear that the coequalizer of the corresponding equivalence relation in **Man** is a single point, and neither in **Man** nor in **Top** the coequalizer is a local homeomorphism, hence, the strong form of axiom (G5) is not satisfied neither in **Man** nor in **Top**.

Another evident counterexample is "irrational" action of $\mathbb{R}$ on torus $S^1 \times S^1$.

So, it is a curious fact, that if one completes both glutoses **Top** and **Man** to universal ultraglutoses (via Main Theorem 10.13), then the "good" orbit spaces always exist inside these completions. Besides, in case of the ultraglutos $\widetilde{\mathbf{Man}}$ the orbit space will always have the tangent bundle as well! This just follows trivially from our Main Theorem 10.13.

And one can pose a question (though rhetorical, yet), whether one can not get an alternative (and more simple) version of "non-commutative differential geometry" of A. Connes (or, at least, part of it), if one develops this theory inside an ultraglutos $\widetilde{\mathbf{Man}}$ in place of the glutos **Man** itself?

6 The 2-category of glutoses

A **morphism** of a glutos $(\mathcal{C}, \mathcal{O})$ into a glutos $(\mathcal{C}', \mathcal{O}')$ is a functor $F: \mathcal{C} \longrightarrow \mathcal{C}'$ between underlying categories, which respects the structures involved, i.e. satisfies the following conditions below:
(MG1) $F(\mathcal{O}) \subset \mathcal{O}'$ and F respects pullbacks of open arrows along arbitrary arrows of $\mathcal{C}$;
(MG2) For any object X of $\mathcal{C}$ the induced functor $F/X: \mathcal{O}/X \longrightarrow \mathcal{O}'/FX$ (which is defined and is left exact, due to (MG1)) is an inverse image of some geometric morphism.

Examples of morphisms of glutoses are: the string of forgetful functors

$$\mathbf{VBun} \longrightarrow \mathbf{Man} \longrightarrow \mathbf{Top} \longrightarrow \mathbf{Set};$$

the tangent functor $T: \mathbf{Man} \longrightarrow \mathbf{VBun}$, as well as the transition to base manifold functor $B: \mathbf{VBun} \longrightarrow \mathbf{Man}$; the functor

$$\mathrm{spec}{:}\mathbf{Schem} \longrightarrow \mathbf{Top}.$$

Besides, all natural functors "inside" a glutos are morphisms of glutoses as shows the following

Proposition 6.3 *For any object X of a glutos $(\mathcal{C}, \mathcal{O})$, the source functor $d_0{:}\mathcal{O}/X \longrightarrow \mathcal{C}$ is a morphism of glutoses; for any arrow $f: X \longrightarrow Y$ of $\mathcal{C}$ the functor $f^*: \mathcal{O}/Y \longrightarrow \mathcal{O}/X$ is a morphism of glutoses.*

Of course, the toposes $\mathcal{O}/X$ and $\mathcal{O}/Y$ above are considered as glutoses via canonical glutos structure of example (0) of Sect.5.

Adding any natural transformations between morphisms of glutoses as 2-arrows one obtains a 2-category **Glut** of glutoses as well as its full subcategory **UGlut** consisting of ultraglutoses.

Now, as is clear enough, the 2-category of toposes imbeds contravariantly to that of glutoses "almost fully" in the sense that any morphism of glutoses $f: \mathcal{E} \longrightarrow \mathcal{E}'$ between toposes $\mathcal{E}$ and $\mathcal{E}'$ decomposes as

$$\mathcal{E} \approx \mathcal{E}/1 \xrightarrow{f/1} \mathcal{E}'/f1 \xrightarrow{d_0} \mathcal{E}' \ ,$$

which means that the "deviation" of a glutos morphism f between toposes from an inverse image of some geometric morphism is just the difference between $f1$ and 1; the imbedding above would be full if one permits for inverse images of geometric morphisms not to respect terminal objects.

Nevertheless, the theory arising is not just generalization of topos theory but, rather, a counterpart to the latter. An essential difference is that **presites** (=categories $\mathcal{C}$, equipped with a pretopology τ [5]) play for glutoses the same role sites play for toposes,

[5]Note only that we will consider sinks in $\mathcal{C}$ and, in particular, coverings of τ as elements of the set $\mathrm{Ob}\mathcal{C} \times \mathcal{P}(\mathrm{Mor}\mathcal{C})$ (where $\mathcal{P}$ stands for power set), rather than indexed families of arrows of $\mathcal{C}$, though in practice indexed families will be used as well, as representing, in an evident way, "real" sinks. The set of all pretopologies on $\mathcal{C}$ forms then a closure system (in the sense of [4]) on the set of all pullbackable sinks of $\mathcal{C}$.

as will be seen in section 10 below[6].

We conclude this section defining a **subglutos** of a glutos $(\mathcal{C}, \mathcal{O})$ as a glutos $(\mathcal{C}', \mathcal{O}')$ such that $\mathcal{C}'$ is a subcategory of $\mathcal{C}$ closed with respect to composition with isomorphisms of $\mathcal{C}$, the set $\mathcal{O}'$ is a subset of $\mathcal{O}$ and, besides, the inclusion functor $\mathcal{C}' \subset \mathcal{C}$ is a morphism of glutoses; the subglutos $(\mathcal{C}', \mathcal{O}')$ is a **full** subglutos of $(\mathcal{C}, \mathcal{O})$ if $\mathcal{C}'$ is a full subcategory of $\mathcal{C}$.

Example: Let $\mathcal{E}$ be a Grothendieck topos (with respect to some universe $\mathcal{U}$); let $\mathcal{E}^-$ be the full subcategory of $\mathcal{E}$ consisting of all pointless objects, i.e. those objects X which have no global sections $1 \longrightarrow X$. Then the pair $(\mathcal{E}^-, \mathcal{E}^-)$ is a full subultraglutos of the ultraglutos $(\mathcal{E}, \mathcal{E})$. If one chooses $\mathcal{E}$ properly, the glutos $(\mathcal{E}^-, \mathcal{E}^-)$ will have no terminal object (see example 0 of section 4).

Counterexample: If $\mathcal{U}$ is any universe containing some infinite set, then the topos $\mathbf{Set}_{\mathcal{U}_f}$ of finite sets is *not* a subglutos of the topos $\mathbf{Set}_{\mathcal{U}}$, because the corresponding inclusion has no right adjoint.

7 $\mathcal{U}$-(ultra)glutoses

For any universe $\mathcal{U}$ there arise a counterpart of Grothendieck toposes (=$\mathcal{U}$-toposes by terminology of [10]). Namely, call an ultraglutos $(\mathcal{C}, \mathcal{O})$ an $\mathcal{U}$-**ultraglutos**, if $\mathcal{C}$ is an $\mathcal{U}$-category, any $\mathcal{O}/X$ is an $\mathcal{U}$-topos and, besides, it satisfies the more strong axiom **(G4$_{\mathcal{U}}$)**, obtained from the axiom **(G4)** by replacing 'finite' by '$\mathcal{U}$-small'.

To define an $\mathcal{U}$-**glutos** one needs as well to strengthen the weak form of the axiom (G5), replacing in p. c) of it 'finitely locally coequalizable' by 'locally coequalizable'. (we will not distinguish in notations axiom **(G5)** for glutoses and for $\mathcal{U}$-glutoses).

By an $\mathcal{U}$-**category** is meant here a category with $\mathcal{U}$-small hom-sets which, besides, is naturally equivalent to some category $\mathcal{C}'$ with $\mathrm{Mor}\mathcal{C}' \subset \mathcal{U}$ (i.e. this definition is stronger than that of [10]).

Remark 7.5 One can show that p.b) of axiom (G3) follows from other axioms for the case of $\mathcal{U}$-glutoses.

Examples (1)–(4) of section 5 above are examples of $\mathcal{U}$-glutoses, which are **not** $\mathcal{U}$-ultraglutoses; another example is a functor category $\mathcal{C}^{\mathcal{D}}$, where $\mathcal{D}$ is $\mathcal{U}$-small and $(\mathcal{C}, \mathcal{O})$ is an $\mathcal{U}$-glutos, if one defines the subcategory $\mathcal{O}'$ in $\mathcal{C}^{\mathcal{D}}$ as follows:

$$\rho\colon F \longrightarrow F'\colon \mathcal{D} \longrightarrow \mathcal{C}$$

belongs to $\mathcal{O}'$ iff for any object D of $\mathcal{D}$ the arrow

$$\rho_D\colon FD \longrightarrow F'D$$

belongs to $\mathcal{O}$. If $\mathcal{C}$ is an $\mathcal{U}$-ultraglutos, then $\mathcal{C}^{\mathcal{D}}$ is an $\mathcal{U}$-ultraglutos as well.

Note that $\mathcal{O}'$ is, generally speaking, bigger than $\mathcal{O}^{\mathcal{D}}$.

[6]Grothendieck topologies are, clearly, non-adequate, because there is no natural subset $\mathcal{O}$ of "open" arrows associated with them. In Appendix B is suggested a generalization of both clopos definition and that of Grothendieck topology (*depending* on generalized clopos structure), which, I beleive, will permit to prove a stronger form of the Main Theorem 10.13 by using generalized Grothendieck topologies in place of pretopologies.

Remark 7.6 Étale structures on an $\mathcal{U}$-topos $\mathcal{E}$ satisfying "collection axiom" in the sense of [12], are particular case of glutos structures on toposes, as one can easily deduce from Corollary 2.3 of [12]. Moreover, for any set Et of étale maps satisfying collection axioms the pair $(\mathcal{E}, Et)$ is a full subglutos of the glutos $(\mathcal{E}, \mathcal{E})$. As to interrelations of glutos structures with étale structures, one can see that any glutos structure $\mathcal{O}$ on an *arbitrary* category $\mathcal{C}$ satisfies all of the conditions (A1)–(A8) of [12] *excepting* conditions (A3) and (A6) (one can easily find counterexamples in the glutos **Top**); and even in the case when $\mathcal{C}$ is an $\mathcal{U}$-topos I have not discovered any special relations (like "descent" and "quotient" axioms of [12]) between glutos structures $\mathcal{O}$ on $\mathcal{C}$ and arbitrary epi's of $\mathcal{C}$.

8 Glueing in glutoses

In this section is studied what kind of pullbacks and colimits exist in $\mathcal{U}$-glutoses or$\mathcal{U}$-ultraglutoses (besides those whose existence is declared by axioms (G1), (G4$_{\mathcal{U}}$) and (G5)).

The general motto here is that in an $\mathcal{U}$-glutos pullback of two arrows exists if it exists locally and that one can glue $\mathcal{U}$-small families of objects along open arrows. The rest of this section is devoted to materialization of this motto into precise statements.

It turns out, in fact, that the corresponding results are valid not only in $\mathcal{U}$-glutoses (resp. in $\mathcal{U}$-ultraglutoses), but, more generally, in any clopos, satisfying the corresponding version of axioms (G4$_{\mathcal{U}}$)-(G5) of section 4 above.

For any set I let ΓI be the category defined as follows. Its set of objects is just the set of all non-empty words of length ≤ 2 in the free monoid $W(I)$ of I-words (which is supposed to be chosen in such a way that I is a subset of $W(I)$ and the canonical map $I \longrightarrow W(I)$ coincides with the inclusion of subsets). The only non-identity arrows of ΓI are arrows

$$i \xleftarrow{d_0^{ij}} ij \xrightarrow{d_1^{ij}} j \quad (i, j \in I);$$

note that d_0^{ii} is to be *different* from d_1^{ii}.

Given a diagram $U: \Gamma I \longrightarrow \mathcal{C}$ we will write U_i, resp. U_{ij}, resp. d_ε^{ij} instead of $U(i)$, resp. $U(ij)$, resp. $U(d_\varepsilon^{ij})$ omitting sometimes superscripts in the latter case; if U has a colimit we will write $U.$ for a colimit object and $\{u_{i}.: U_i \longrightarrow U.\}_{i \in I}$ for a colimit cone.

Call a diagram $U: \Gamma I \longrightarrow \mathcal{C}$ **glueing data** or a **gluon** if the following "cocicle conditions" are satisfied:

(GD1) For any $i, j \in I$ the pair (d_0^{ij}, d_1^{ij}) is a mono source in $\mathcal{C}$;

(GD2) For any $i, j \in I$ there exists an arrow $\tau_{ij}: U_{ij} \longrightarrow U_{ji}$ of $\mathcal{C}$ such that the equalities

$$d_0^{ji}\tau_{ij} = d_1^{ij} \quad \text{and} \quad d_1^{ji}\tau_{ij} = d_0^{ij};$$

are valid (it then follows from (GD1) that $\tau_{ij}\tau_{ji} = \mathrm{Id}$);

(GD3) For any $i \in I$ there exists an arrow $s_i: U_i \longrightarrow U_{ii}$ of $\mathcal{C}$ which is right inverse to both d_0^{ii} and d_1^{ii};

(GD4) For any word ijk of length 3 in $W(I)$ there exists an object U_{ijk} of $\mathcal{C}$ and the arrows $p_0: U_{ijk} \longrightarrow U_{ij}$, $p: U_{ijk} \longrightarrow U_{ik}$ and $p_1: U_{ijk} \longrightarrow U_{jk}$ such that all three squares of

the diagram

$$
\begin{tikzcd}
& U_{ijk} \arrow[dl, "p_0"'] \arrow[d, "p"] \arrow[dr, "p_1"] & \\
U_{ij} \arrow[d, "d_0"'] \arrow[dr, "d_1" near start] & U_{ik} \arrow[dl, "d_0" near start] \arrow[dr, "d_1"' near start] & U_{jk} \arrow[dl, "d_0"' near start] \arrow[d, "d_1"] \\
U_i & U_j & U_k
\end{tikzcd}
\tag{5}
$$

are pullbacks (we will write further p_0^{ijk}, etc. instead of p_0 in case of necessity).

It follows from the latter condition the existence of isomorphisms $\theta_{ijk}: U_{ijk} \longrightarrow U_{jki}$ which agree with projections p_0, p_1 and p "twisted" by iso's τ_{ij} and satisfy the "cocicle conditions" arising both in algebraic and differential geometry in processes of glueing of schemes, resp. manifolds along open subschemes, resp. open submanifolds.

One can see, on the other hand, that if the index set I consists of just one element the definition above reproduces the definition of an equivalence relation.

More generally, for any glueing data $U: \Gamma I \longrightarrow \mathcal{C}$ and any $i \in I$ the pair $d_0^{ii}, d_1^{ii}: U_{ii} \rightrightarrows U_i$ is, evidently, an equivalence relation.

Any family $\{u_i: U_i \longrightarrow X\}_{i \in I}$ of pullbackable arrows defines, canonically, some glueing functor if one sets $U_{ij} \approx U_i \prod_X U_j$, whereas for d_ε^{ij} one chooses the corresponding pullback projections.

Call a diagram in a clopos (resp. in a glutos) **clopen** (resp. **open**) if any arrow of this diagram is clopen (resp. open). Call a clopen gluon $U: \Gamma I \longrightarrow \mathcal{C}$ in a clopos $(\mathcal{C}, \mathcal{O})$ **(finitely) locally $\mathcal{O}$-coequalizable** if (I is finite and) for any $i \in I$ the equivalence relation $d_0^{ii}, d_1^{ii}: U_{ii} \rightrightarrows U_i$ is (finitely) locally $\mathcal{O}$-coequalizable.

One sees immediately that for any clopen gluon in a clopos, morphisms τ_{ij}, p_0, p_1 and p of (GD2), (GD4) are clopen. As to (the only by (GD1)) arrows s_i of (GD3), they also are clopen if the clopos satisfies conditions (G4$_\mathcal{U}$)-(G5) as one can see from the following

Proposition 8.4 *Suppose that a clopos* $(\mathcal{C}, \mathcal{O})$ *satisfies conditions* (G4$_\mathcal{U}$)-(G5) *in the definition of* $\mathcal{U}$*-ultraglutoses* (resp. *of* $\mathcal{U}$*-glutoses*). *Then:*
(G4$_\mathcal{U}$ + 5) *Any* $\mathcal{U}$*-small clopen gluon* (resp. *any* $\mathcal{U}$*-small clopen locally* $\mathcal{O}$*-coequalizable gluon*)[7] $U: \Gamma I \longrightarrow \mathcal{C}$ *has a universal colimit* $U.$ *which, besides, is effective in the sense that for any* $i, j \in I$ *the isomorphism*

$$U_{ij} \approx U_i \prod_{U.} U_j$$

holds. The colimit cone $\{U_i \longrightarrow U.\}_{i \in I}$ *consists of clopen arrows.*

Indications to the proof. Consider the diagram

$$d_0, d_1: \coprod_{i,j \in I} U_{ij} \rightrightarrows \coprod_{i \in I} U_i, \tag{6}$$

[7]Note that any clopen gluon every arrow of which is mono, is locally $\mathcal{O}$-coequalizable automatically, because corresponding equivalence relations are trivial in this case: both d_0 and d_1 are iso's.

where, say, d_0 is defined as $(\iota_i d_0^{ij})_{i,j\in I}$ with $\iota_i\colon U_i \longrightarrow \coprod_{i\in I} U_i$ being the canonical coproduct injection arrows[8]. One is to prove that the diagram above is an open equivalence relation (resp. is an open $\mathcal{O}$-coequalizable equivalence relation); then it will follow trivially that the coequalizer

$$q\colon \coprod_{i\in I} U_i \longrightarrow U.$$

of this diagram (existing by $(\mathbf{G5}_{\mathcal{U}})$) reproduces the colimit cone of the original gluon U if one sets $u_i. = q\iota_i$.

Note first that the families $\{\tau_{ij}\}_{i,j\in I}$ and $\{s_i\}_{i\in I}$ of (GD2) and (GD3) permit one to build in a natural way the arrows $\tau\colon \coprod U_{ij} \longrightarrow \coprod U_{ij}$ and $s\colon \coprod U_i \longrightarrow \coprod U_{ij}$; the verification of the fact that these arrows satisfy (GD2), resp.(GD3), is straightforward.

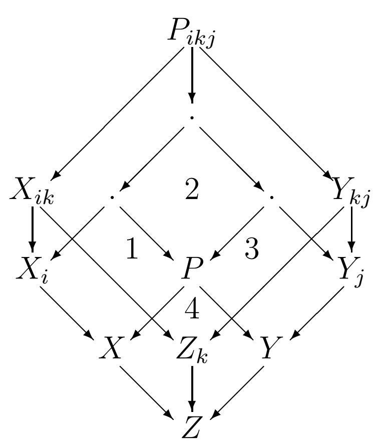
Similarly, one can construct three arrows p_0, p_1 and p from $\coprod U_{ijk}$ to $\coprod U_{ij}$; e.g., the arrow $p_0\colon \coprod U_{ijk} \longrightarrow \coprod U_{ij}$ is defined to be the colimit arrow

$$(U_{ijk} \xrightarrow{p_0^{ijk}} U_{ij} \xrightarrow{\iota_{ij}} \coprod U_{ij})_{i,j,k\in I} \quad ,$$

where ι_{ij} are canonical coproduct injection arrows.

In proving (GD4) for the diagram 6 above the following useful lemma can be used, which states that a square is a pullback if it is a pullback locally:

Lemma 8.5 *Let $f\colon X \longrightarrow Z$ and $g\colon Y \longrightarrow Z$ be arrows of a category $\mathcal{C}$; let $\{x_i\colon X_i \longrightarrow X\}_{i\in I}$, $\{y_j\colon Y_j \longrightarrow Y\}_{j\in J}$ and $\{z_k\colon Z_k \longrightarrow Z\}_{k\in K}$ are universal effective epi families; let, further, for any $i \in I$, $j \in J$ and $k \in K$ a diagram*



(7)

is given such that four side squares of it as well as the "floor" squares 1, 2 and 3 are pullbacks. Then the square 4 is a pullback iff for any $i \in I$, $j \in J$ and $k \in K$ the "ceiling" square is a pullback.

Applying this lemma to the case $X = Y = \coprod U_{ij}$, $Z = \coprod U_i$ and $P = \coprod U_{ijk}$ with the corresponding universal effective epi families being $\{\iota_{ij}\colon U_{ij} \longrightarrow \coprod U_{ij}\}_{i,j\in I}$, etc., one

[8] We use parentheses instead of braces in order to distinguish between *families* of arrows and a single (co)limit arrow determined by the corresponding family.

obtains after simple diagram chase just squares of the diagram (5) as "ceilings" of the diagram (7) above, which proves (GD4) for the diagram (6).

At last, to prove (GD1) for the diagram (6) consider a pair of arrows $f, g\colon X \rightrightarrows \coprod U_{ij}$ such that both $d_0 f = d_0 g$ and $d_1 f = d_1 g$. Pulling a covering $\{\iota_i\colon U_i \longrightarrow \coprod U_i\}_{i\in I}$ along $d_0 f = d_0 g$, resp. along $d_1 f = d_1 g$, one obtains two universal effective epi families $\{v_i\colon V_i \longrightarrow X\}_{i\in I}$ and $\{v'_j\colon V_j \longrightarrow X\}_{j\in I}$ such that f agree with g on elements of the "intersection" universal effective epi family $\{V_i \coprod_X V'_j \longrightarrow X\}_{i,j\in I}$, which implies that $f = g$. ■

The following proposition describes sufficient conditions of existence of pullbacks in cloposes satisfying (G4$_{\mathcal{U}}$)-(G5).

Proposition 8.6 *Let a clopos* $(\mathcal{C}, \mathcal{O})$ *satisfies conditions* (G4$_{\mathcal{U}}$) *as well as the strong axiom* (G5) *or the combination of the weak axiom* (G5) *with the axiom* (G5P)*. Let* $f\colon X \longrightarrow Z$ *and* $g\colon Y \longrightarrow Z$ *be arrows of* $\mathcal{C}$ *such that for some* $\mathcal{U}$*-small epi families of clopen arrows* $\{X_i \longrightarrow X\}_{i\in I}$, $\{Y_j \longrightarrow Y\}_{j\in J}$ *and* $\{Z_k \longrightarrow Z\}_{k\in K}$ *there exists, for any* $i \in I$, $j \in J$ *and* $k \in K$, *the pullback* $X_{ik} \prod_{Z_k} Y_{kj}$, *where, by definition,* $X_{ik} = X_i \prod_Z Z_k$ *and* $Y_{kj} = Z_k \prod_Z Y_j$*. Then there exists the pullback of* f *and* g*. If, besides,* $(\mathcal{C}, \mathcal{O})$ *is an* $\mathcal{U}$*-glutos, then the* $\mathcal{U}$*-smallness condition for families above can be omitted.*

The archetype of the proof of Prop. 8.6 is contained, for example, in the proof of existence of pullbacks of Grothendieck schemes (see, e.g., [13]).

Proposition 8.4 permits one to equip, canonically, any $\mathcal{U}$-glutos $(\mathcal{C}, \mathcal{O})$ with a structure of a presite, but, before going into details, one needs to give some necessary definitions and to state some elementary properties of presites.

9 Presites

Define first, for a presite $(\mathcal{C}, \tau)$ the set $\mathcal{O}_\tau$ of arrows of $\mathcal{C}$ as consisting of just those arrows which belong to some covering of τ. The set $\mathcal{O}_\tau$ satisfies condition (G1) above (so that its elements will be referred to as clopen or open) and one can define **morphisms** between presites $(\mathcal{C}, \tau)$ and $(\mathcal{C}', \tau')$ as just those functors between underlying categories which respect coverings and satisfy condition (MG1) above (with $\mathcal{O}$ replaced by $\mathcal{O}_\tau$, idem for $\mathcal{O}'$). We will call such functors **continuous** (this definition is stronger than the corresponding definition in [10] making emphasis on topologies and sites).

If $F\colon \mathcal{C} \longrightarrow \mathcal{C}'$ is a functor and τ' is a pretopology on $\mathcal{C}'$, then a pretopology τ on $\mathcal{C}$ will be called **induced** by τ' along F iff for any sink S in $\mathcal{C}$ the condition $FS \in \tau'$ is equivalent to $S \in \tau$; if such τ exists it is the biggest pretopology on $\mathcal{C}$ making F continuous.

Proposition 9.7 *For any presite* $(\mathcal{C}, \tau)$ *and any object* X *of* $\mathcal{C}$ *there exists the pretopology on* $\mathcal{O}_\tau/X$ *induced by* τ *along the "source" functor* $d_0\colon \mathcal{O}_\tau/X \longrightarrow \mathcal{C}$.

The category $\mathcal{O}_\tau/X$ will be considered, canonocally, to be equipped with this presite structure; then:

Proposition 9.8 *For any arrow* $f\colon X \longrightarrow Y$ *of* $\mathcal{C}$ *the induced functor* $f^*\colon \mathcal{O}_\tau/Y \longrightarrow \mathcal{O}_\tau/X$ *is continuous.*

Now, a presite $(\mathcal{C}, \tau)$ will be called an $\mathcal{U}$-**presite** if $\mathcal{C}$ is an $\mathcal{U}$-category and, besides, the following condition is satisfied (existence of local sets of topological generators):
(LTG$_{\mathcal{U}}$) For any object X of $\mathcal{C}$ there exists an $\mathcal{U}$-small subset G_X of objects of $\mathcal{C}$ such that for any clopen arrow $u: U \longrightarrow X$ of $\mathcal{C}$ there exists a covering $\{u_i: U_i \longrightarrow U\}_{i \in I}$ such that any U_i belongs to G_X. This condition is just equivalent to saying that any $\mathcal{O}_\tau / X$, considered as a site, is an $\mathcal{U}$-site in the sense of [10].

Remark 9.7 It is convenient to include in the definition of a pretopology the following condition (completeness property):
(PT4) If $\langle X, S \rangle$ is a sink of $\mathcal{C}$ such that $S \subset \mathcal{O}_\tau$ (such sinks will be called **(cl)open**) and there exists a refinement of $\langle X, S \rangle$ which is a covering of X then $\langle X, S \rangle$ itself is a covering. Here a sink $\langle X, S' \rangle$ is said to be a **refinement** of $\langle X, S \rangle$ if for any $s' \in S'$ there exists $s \in S$ such that s' factors through s.

Intersection of pretopologies satisfying (PT4) satisfies (PT4) itself; if τ satisfies (PT4) then a pretopology induced by τ along any functor satisfies (PT4) as well; besides, the completion of a pretopology τ satisfying ordinary conditions (PT1)–(PT3) of [10] to that satisfying (PT4), does not change neither the set $\mathcal{O}_\tau$, neither the associated Grothendieck topology[9],nor the universal glutos of Theorem 10.13 below. That is why from now on 'pretopology' will mean 'pretopology satisfying (PT4)' with similar change in the meaning of 'presite'.

Remark 9.8 If one looks at the definition of elementary glutos, a natural question can arise: what will happen if one "iterates" the theory of glutoses replacing, roughly, in axioms (G1)–(G5) "topos" by "glutos"? The answer is that the theory of elementary glutoses is stable by this iteration, i.e., no new "weaker" theory will arise.

In more details, defining, in an evident way, morphisms of cloposes as well as clopos structure induced along a functor, one obtains that for any object X of any clopos $(\mathcal{C}, \mathcal{O})$ the category $\mathcal{O}/X$ has a clopos structure $\mathcal{O}_X$ induced along the functor d_0: arrows of $\mathcal{O}_X$ are all commutative triangles (i.e., arrows of $\mathcal{O}/X$) all three arrows of which belong to $\mathcal{O}$. Counterparts of Props. 9.7 and 9.8 are valid for cloposes as well as the following result:

Proposition 9.9 *For any clopen arrow $U \xrightarrow{u} X$ in a clopos $(\mathcal{C}, \mathcal{O})$ the functor*

$$d_0/(U \xrightarrow{u} X): \mathcal{O}_X/(U \xrightarrow{u} X) \longrightarrow \mathcal{O}/U$$

is a natural equivalence.

Now, if one *removes* the axiom (G2) and one replaces in axiom (G3) "topos" by "glutos", resp. "inverse image of geometric morphism" by "morphism of glutoses", interpreting, simultaneously, $\mathcal{O}/X$, etc. not simply as categories but as cloposes via induced structure, then one arrives to an elementary theory which turns out to be not weaker, but equivalent to the theory of elementary glutoses. This just follows from Prop. 9.9.

[9]This completion is, in fact, the biggest pretopology among those having both the same set of open arrows and the same associated Grothendieck topology as τ has.

10 $\mathcal{U}$-glutoses as $\mathcal{U}$-presites

Returning again to $\mathcal{U}$-glutoses, one has:

Proposition 10.10 *Let* $(\mathcal{C}, \mathcal{O})$ *be an* $\mathcal{U}$*-glutos. Then:*
(G6$_\mathcal{U}$) *All epi sinks in* $\mathcal{C}$ *with elements in* $\mathcal{O}$ *are universal effective and, hence, form some pretopology on* $\mathcal{C}$ (denoted further $\tau_\mathcal{O}$). This pretopology is **subcanonical** (i.e. the associated topology is subcanonical);
(G7$_\mathcal{U}$) *The presite* $(\mathcal{C}, \tau_\mathcal{O})$ *is an* $\mathcal{U}$*-presite;*
(G8$_\mathcal{U}$) *Any sink* $\langle X, S\rangle$ *with* $S \subset \mathcal{O}$ *factors as a covering of* $\tau_\mathcal{O}$ *followed by an open mono.*
(G9$_\mathcal{U}$) (local character of open arrows) *If for* $u\colon U \longrightarrow X$ *there exists a covering* $\{u_i\colon U_i \longrightarrow U\}_{i\in I}$ *of* $\tau_\mathcal{O}$ *such that for any* $i \in I$ *the arrow* uu_i *is open, then* u *itself is open.*
(G10) *For any object* X *of* $\mathcal{C}$ *the pretopology on* $\mathcal{C}/X$ *induced by* $\tau_\mathcal{O}$ *is the canonical pretopology of the topos* $\mathcal{C}/X$ (i.e. coverings of it are all epi sinks); *moreover, the source functor* $d_0\colon \mathcal{C}/X \longrightarrow \mathcal{C}$ *respects both coequalizers of equivalence relations and* $\mathcal{U}$*-small coproducts.*

Remark 10.9 For an elementary glutos $(\mathcal{C}, \mathcal{O})$ let $\tau_\mathcal{O}$ be the set of all open sinks having some finite open epi refinement. Then one has: **(G6)** $\tau_\mathcal{O}$ is a subcanonical pretopology on $\mathcal{C}$; the counterparts of (G8$_\mathcal{U}$) and (G9$_\mathcal{U}$) are valid as well if one replaces in (G8$_\mathcal{U}$) 'Any sink' by 'Any finite sink'.

The following proposition is a counterpart of Giraud theorem :

Proposition 10.11 *A pair* $(\mathcal{C}, \mathcal{O})$ *is an* $\mathcal{U}$*-glutos* (resp. *an* $\mathcal{U}$*-ultraglutos*) *iff it satisfies conditions* (G1)–(G2), (G4$_\mathcal{U}$), *weak axiom* (G5) (resp., *strong axiom* (G5) *and axiom* (G5P)), (G6$_\mathcal{U}$)-(G7$_\mathcal{U}$) *above* (conditions (G4$_\mathcal{U}$) (G5) and (G5P) can be replaced by weak (resp. strong) form of condition (G4$_\mathcal{U}+5$) and condition (G9$_\mathcal{U}$)).

Now, a map $(\mathcal{C}, \mathcal{O}) \mapsto (\mathcal{C}, \tau_\mathcal{O})$ continues to the 2-functor imbedding fully $\mathcal{U}$-glutoses into $\mathcal{U}$-presites, as shows the following

Proposition 10.12 *Let* $(\mathcal{C}, \mathcal{O})$ *and* $(\mathcal{C}', \mathcal{O}')$ *be* $\mathcal{U}$*-glutoses and* $F\colon \mathcal{C} \longrightarrow \mathcal{C}'$ *be a functor. Then* F *is morphism of glutoses iff it is continuous w.r.t. pretopologies* $\tau_\mathcal{O}$ *and* $\tau_{\mathcal{O}'}$*.*

In other words, the 2-category $\mathbf{Glut}_\mathcal{U}$ of $\mathcal{U}$-glutoses may be considered as a full 2-subcategory of the 2-category $\mathbf{Psite}_\mathcal{U}$ of $\mathcal{U}$-presites

$$\mathbf{Glut}_\mathcal{U} \hookrightarrow \mathbf{Psite}_\mathcal{U}. \tag{8}$$

Remark 10.10 Let **Glut.**, be the full 2-subcategory of **Glut**, containing any glutos which is $\mathcal{U}$-glutos for some universe $\mathcal{U}$. Let the 2-category **Psite.** be defined similarly. The above inclusion functor continues to the inclusion functor

$$\mathbf{Glut.} \hookrightarrow \mathbf{Psite.}\ ,$$

but the counterexample

$$\mathbf{Set}_{\mathcal{U}_f} \hookrightarrow \mathbf{Set}_\mathcal{U}$$

of continuous functor which is not a morphism of glutoses (see the end of section 6) shows that this inclusion is *not* full.

The main author's result states that *the 2-category* $\mathbf{Glut}_{\mathcal{U}}$ *of* $\mathcal{U}$*-glutoses is reflective in the 2-category* $\mathbf{Psite}_{\mathcal{U}}$ *of* $\mathcal{U}$*-presites, whereas the 2-category* $\mathbf{UGlut}_{\mathcal{U}}$ *of* $\mathcal{U}$*-ultraglutoses is reflective in the 2-category* $\mathbf{Glut}_{\mathcal{U}}$. In more details:

Theorem 10.13 (a) *For any* $\mathcal{U}$*-presite* $\mathcal{C}$ *there exists an* $\mathcal{U}$*-(ultra)glutos* $\widetilde{\mathcal{C}}$ *together with an arrow* $Y_{\mathcal{C}}\colon \mathcal{C} \longrightarrow \widetilde{\mathcal{C}}$ *which is universal in the sense that for any* $\mathcal{U}$*-glutos* $\mathcal{D}$ *the arrow*

$$[Y_{\mathcal{C}}, \mathcal{D}]\colon [\widetilde{\mathcal{C}}, \mathcal{D}] \longrightarrow [\mathcal{C}, \mathcal{D}] \quad (\sigma \mapsto \sigma Y_{\mathcal{C}}) \tag{9}$$

is a natural equivalence having right inverse;
(b) *the arrow* $Y_{\mathcal{C}}$ (or, rather, the underlying functor) *can always be chosen to be injective on objects of* $\mathcal{C}$*; if the pretopology* τ *of* $\mathcal{C}$ *is subcanonical, then the arrow* $Y_{\mathcal{C}}$ *is fully faithful;*
(c) *The universal arrow* $Y_{\mathcal{C}}$ *reflects open coverings* (see sect. 12 below for the definition)*; for every object* X *of* $\widetilde{\mathcal{C}}$ *the set* G_X *of topological generators of* X (see ($\mathrm{LTG}_{\mathcal{U}}$) in sect. 9 above) *can be chosen to belong to the set* $Y_{\mathcal{C}}(\mathcal{C})$*;*
(d) *Besides, if the underlying category of* $\mathcal{C}$ (denoted further $\mathcal{C}$, by abuse of notation) *is* $\mathcal{U}$*-cocomplete and the pretopology of* $\mathcal{C}$ *is subcanonical, then the functor* $Y_{\mathcal{C}}$ *has left adjoint* $\Gamma\colon \widetilde{\mathcal{C}} \longrightarrow \mathcal{C}$ (the **global sections functor**). Note that Γ need not be continuous.

The proof of Th.10.13 is sketched in Appendix A.

Remark 10.11 It follows from Appendix A that the 2-category of subcanonical $\mathcal{U}$-presites is as well reflective in $\mathbf{Psite}_{\mathcal{U}}$, i.e. the universal arrow $Y_{\mathcal{C}}$ decomposes as

$$\mathcal{C} \xrightarrow{Y'_{\mathcal{C}}} \mathcal{C}_{sub} \xrightarrow{Y_{\mathcal{C}_{sub}}} \widetilde{\mathcal{C}_{sub}},$$

where $\mathcal{C}_{sub}$ is a universal subcanonical $\mathcal{U}$-presite for $\mathcal{C}$.

Now choosing for every $\mathcal{U}$-presite $\mathcal{C}$ some universal arrow $Y_{\mathcal{C}}$ and choosing for every pair $\mathcal{C}$, $\mathcal{D}$ as in p.(a) of Theorem 10.13 some arrow

$$I_{\mathcal{C}\mathcal{D}}\colon [\mathcal{C}, \mathcal{D}] \longrightarrow [\widetilde{\mathcal{C}}, \mathcal{D}]$$

right inverse to the arrow (9), we will obtain for every pair $\mathcal{C}$, $\mathcal{C}'$ of $\mathcal{U}$-presites some functor

$$[\mathcal{C}, \mathcal{C}'] \xrightarrow{\sim} [\widetilde{\mathcal{C}}, \widetilde{\mathcal{C}'}] \quad (\sigma \mapsto \widetilde{\sigma}) \ ,$$

defined by $\widetilde{\sigma} := I_{\mathcal{C}\widetilde{\mathcal{C}'}}(Y_{\mathcal{C}'}\sigma)$ on 2-arrows from $[\mathcal{C}, \mathcal{C}']$.
The correspondences $\sigma \mapsto \widetilde{\sigma}$ just defined are incorporating together to give some pseudofunctor (see [10],[9])

$$\mathbf{Psite}_{\mathcal{U}} \xrightarrow{\sim} \mathbf{Glut}_{\mathcal{U}} \ ,$$

left quasiadjoint to the inclusion 2-functor (8); it differs from a 2-functor by some "twisting by a cocicle" $\sigma(F, F')\colon \widetilde{F'}\widetilde{F} \longrightarrow \widetilde{(F'F)}$. The following theorem shows that this cocicle can be killed.

Theorem 10.14 *The correspondences* $\mathcal{C} \mapsto \widetilde{\mathcal{C}}$ *and* $F \mapsto \widetilde{F}$ *can be chosen in such a way that* $\widetilde{F'}\widetilde{F} = \widetilde{(F'F)}$ *for every composable pair* F *and* F' *of morphisms of* $\mathcal{U}$*-presites.*

Corollary 10.15 *If $F\colon \mathcal{C} \longrightarrow \mathcal{C}'$ and $G\colon \mathcal{C}' \longrightarrow \mathcal{C}$ are continuous functors between $\mathcal{U}$-presites such that F is left adjoint to G then $\widetilde{F}$ is left adjoint to $\widetilde{G}$.*

Exactness properties of universal arrows are described by the following

Proposition 10.16 (a) *For any $\mathcal{U}$-presite $\mathcal{C}$ the universal arrow $Y_{\mathcal{C}}\colon \mathcal{C} \longrightarrow \widetilde{\mathcal{C}}$ respects all $\mathcal{U}$-limits existing in $\mathcal{C}$;*
(b) *If $\mathcal{C}$ has pullbacks, resp. products, resp. finite limits, then so does $\widetilde{\mathcal{C}}$;*
(c) *Let $F\colon \mathcal{C} \longrightarrow \mathcal{C}'$ be morphism of presites and $\mathcal{C}$ has products, resp. pullbacks, resp. finite limits, which are, besides, respected by the functor F. Then the functor $\widetilde{F}\colon \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{C}}'$ respects products, resp. pullbacks, resp. finite limits.*

Theorem 10.13 and Prop.10.16 show that glutoses are "invariants" of presites in just the same way as toposes are "invariants" of sites. The universal arrow for an $\mathcal{U}$-presite is, clearly, a counterpart of topos-theoretic "sheafified Yoneda functor" $Y\colon \mathcal{S} \longrightarrow \mathrm{Sh}\mathcal{S}$ associating to any $\mathcal{U}$-site $\mathcal{S}$ the topos of $\mathbf{Set}_{\mathcal{U}}$-valued sheaves on it. In many familiar cases of $\mathcal{U}$-presites $\mathcal{C}$ (see examples of universal arrows below), the corresponding *site* is not an $\mathcal{U}$-site, which means that the topos of sheaves on this site exists in some *higher* universe only. At the same time, the glutos $\widetilde{\mathcal{C}}$ associated with the presite $\mathcal{C}$ exists in the *same* universe $\mathcal{U}$, where $\mathcal{C}$ is contained. Nevertheless, when both constructions exist, they sometimes coincide as shows the following

Proposition 10.17 *Let $\mathcal{C}$ be $\mathcal{U}$-small and finitely complete. Let a pretopology τ on $\mathcal{C}$ be given, such that any arrow of $\mathcal{C}$ is clopen. Then the universal glutos $\widetilde{\mathcal{C}}$ coincides with the topos of sheaves* $\mathrm{Sh}\mathcal{C}$ *up to natural equivalence of categories. The similar is true for universal arrows.*

For example, the glutos constructed from a topological space is the same thing as the topos of sheaves on it; the same is true for a complete Heyting algebra (equipped with the canonical (pre-)topology).

Many familiar examples of constructing categories out of "simpler ones" by means of "charts and atlases" routine are just concrete realizations of universal arrows of Theorem 10.13: imbedding of smooth euclidean regions into the category of smooth manifolds, imbedding of trivial vector bundles into the category of locally trivial ones, as well as the functor

$$\mathrm{Spec}\colon\mathbf{Ring}^{op} \longrightarrow \mathbf{Schem}.$$

Note that in this example the global sections functor of p.(d) of Theorem 10.13 exists and is the same thing as the ordinary global sections functor on **Schem**, which justifies the use of the name "global sections functor" in the general case.

The latter example opens up a new approach to "universal algebraic geometry", alternative to that of M.Coste [5] (based on Hakim's theorem): given some locally finitely presentable category (see [8]) $\mathcal{C}$ together with some pretopology τ on its dual category, turning $\mathcal{C}^{op}$ into an $\mathcal{U}$-presite, the category of **schemes over** $\mathcal{C}$ and the corresponding functor Spec can be *defined* to be, respectively, the glutos associated with the presite $(\mathcal{C}^{op}, \tau)$ and the universal arrow for it.

For example, if one chooses the étale pretopology on the category dual to that of commutative rings instead of Zariski pretopology, one obtains the category **ESchem**, which may be called the category of **étale schemes**; the functor

$$\mathbf{Schem} \longrightarrow \mathbf{ESchem}, \tag{10}$$

provided by Theorem 10.13, fully imbeds the category of schemes into that of étale schemes in such a way that for any scheme X the topos of sheaves over X with respect to étale pretopology on Et/X imbeds into **ESchem** via

$$\mathrm{Sh}(Et/X) \hookrightarrow \mathbf{ESchem}/X \xrightarrow{d_0} \mathbf{ESchem}.$$

Another application is concerned with "non-commutative algebraic geometry": Theorem 10.13 gives general non-commutative schemes glued out of non-commutative affine schemes of P.M.Cohn [3].

A class of pretopologies on duals to locally finitely presentable categories especially suitable for "universal algebraic geometry" will be considered elsewhere.

11 $\mathcal{M}$-presites and SG-glutoses.

In this section some natural endo-2-functors are constructed on the 2-category **Psite** of all presites; recall,that $\mathcal{U}$-glutoses are considered as presites via Props. 10.10 and 10.12. Besides, a class of $\mathcal{U}$-glutoses which occur particularly often in practice is studied in more details.

Let $\mathcal{P}$ be some property of an arrow of a presite $\mathcal{C}$. We say that an arrow $f\colon X \longrightarrow Y$ of $\mathcal{C}$ **locally satisfies** $\mathcal{P}$ or **is locally** $\mathcal{P}$, if there exists a covering $\{u_i\colon U_i \longrightarrow X\}_{i\in I}$ such that for every $i \in I$ the arrow fu_i satisfies $\mathcal{P}$. We will use further this metadefinition for the case when the property $\mathcal{P}$ is either "f is (cl)open" or "f is an (cl)open mono" getting the properties "f is locally (cl)open" or "f is locally an (cl)open mono" (note yet that an f which locally is a clopen mono need not to be neither clopen nor mono). One can easily verify that the set of all *pullbackable* locally open arrows of any presite is closed both with respect to compositions and arbitrary pullbacks.

For example, the property ($\mathrm{G9}_{\mathcal{U}}$) of glutoses (see section 10) can be reformulated in this terms as follows: every locally open arrow in a glutos is open.

Let τ be a pretopology on a category $\mathcal{C}$. Define the pretopology $\mathcal{M}\tau$, resp. $\mathcal{L}\tau$, resp. $\mathrm{SG}(\tau)$ on the category $\mathcal{C}$ as follows: coverings of $\mathcal{M}\tau$ are all coverings of τ consisting of mono's; coverings of $\mathcal{L}\tau$ are all sinks of τ consisting of pullbackable locally open arrows of $\mathcal{C}$ and having a refinement belonging to a pretopology τ (the latter definition is correct because pullbackable locally open arrows form a clopos structure as stated above); at last let $\mathrm{SG}(\tau) = (\mathcal{L}\mathcal{M}\tau) \cap \tau$. One has, evidently, the following inclusions:

$$\mathcal{M}\tau \subset \mathrm{SG}(\tau) \subset \tau \subset \mathcal{L}\tau \ .$$

The operations $\mathcal{M}$, $\mathcal{L}$ and SG can be continued to the endo-2-functors (denoted by the same symbols) on the 2-category **Psite**, whereas the chain of inclusions above produce the chain of 2-functor morphisms

$$\mathcal{M} \hookrightarrow \mathrm{SG} \hookrightarrow \mathrm{Id}_{\mathbf{Psite}} \hookrightarrow \mathcal{L} \ , \tag{11}$$

which go to identity 2-functor morphisms when being composed with the neglecting 2-functor from **Psite** to the 2-category **Cat** of all categories.

It is evident that for any $\mathcal{U}$-presite $\mathcal{C}$ the presite $\mathcal{L}\mathcal{C}$ is an $\mathcal{U}$-presite (but the author do not know at present whether or not the same is true for $\mathcal{M}\mathcal{C}$ and $\mathrm{SG}\mathcal{C}$). An evident fact that the Grothendieck *topologies* generated by pretopologies of $\mathcal{C}$ and of $\mathcal{L}\mathcal{C}$ coincide, imply, together with the construction of universal glutoses out of the corresponding category of "big" sheaves (see Appendix A), the following

Proposition 11.18 *For any $\mathcal{U}$-presite $\mathcal{C}$ there is a canonical natural equivalence $\widetilde{\mathcal{C}} \approx \widetilde{\mathcal{L}\mathcal{C}}$; in more details, the composition arrow*

$$\mathcal{C} \hookrightarrow \mathcal{L}\mathcal{C} \xrightarrow{Y_{\mathcal{L}\mathcal{C}}} \widetilde{\mathcal{L}\mathcal{C}}$$

is a universal arrow for $\mathcal{C}$.

The following proposition describes the monoid of endo-2-functors generated by $\mathcal{M}$, SG and $\mathcal{L}$.

Proposition 11.19 *The 2-functors $\mathcal{M}$,* SG *and $\mathcal{L}$ satisfy the following algebraic relations:*

$$\begin{array}{rrr} \mathcal{M}^2 = \mathcal{M}, & \mathrm{SG}^2 = \mathrm{SG}, & \mathcal{L}^2 = \mathcal{L}, \\ (\mathcal{M}\mathcal{L})^2 = \mathcal{M}\mathcal{L}, & (\mathcal{L}\mathcal{M})^2 = \mathcal{L}\mathcal{M}, & \mathrm{SG}\mathcal{L} = \mathcal{L}\mathcal{M}\mathcal{L}, \\ \mathcal{L}\mathrm{SG} = \mathcal{L}\mathcal{M}, & \mathrm{SG}\mathcal{M} = \mathcal{M}, & \mathcal{M}\mathrm{SG} = \mathcal{M}. \end{array}$$

The only relation amongst those above, whose verification uses drawing of some diagrams is that stating the idempotence of the functor $\mathcal{L}$.

The first three relations of proposition 11.19 together with the universality properties of functor morphisms (11) imply that both the full 2-subcategory of presites stable by $\mathcal{M}$ and of presites stable by SG are coreflective in **Psite**, whereas the full 2-subcategory of presites stable by $\mathcal{L}$ is reflective in **Psite**.

A presite stable by $\mathcal{M}$, resp, by SG, resp. by $\mathcal{L}$ will be called an $\mathcal{M}$ **-presite**, resp. an SG**-presite**, resp. an $\mathcal{L}$ **-presite**. In other words, a presite $\mathcal{C}$ is an $\mathcal{M}$-presite iff any covering of it consists of mono's; it is an SG-presite iff any clopen arrow of it is locally a clopen mono; it is an $\mathcal{L}$-presite iff any arrow of it which is locally clopen is clopen.

In particular, any $\mathcal{U}$-glutos is an $\mathcal{L}$-presite; an $\mathcal{U}$-glutos $(\mathcal{C}, \mathcal{O})$ is an SG-presite iff for any object X of $\mathcal{C}$ the topos $\mathcal{O}/X$ is an SG-topos as defined in [11], which justifies the name "SG-glutos" for the general case.

Glutoses of examples (1)–(4) of section 5 above are SG-glutoses, as well as $\mathcal{C}^{\mathcal{D}}$ when $\mathcal{C}$ is an SG-glutos; the glutos of étale schemes is not an SG-glutos. Any $\mathcal{U}$-topos has, canonically, a structure of an SG-glutos, if one defines open arrows as just those arrows $u\colon U \longrightarrow X$ which locally are mono (here "locally" is, of course, with respect to canonical pretopology of the topos). In fact, the latter example can be generalized, as shows the following proposition, easily deduced from "Giraud theorem" 10.11 and the fact that $\mathcal{U}$-toposes are locally $\mathcal{U}$-small (see p.251 of [10]):

Proposition 11.20 *For any* $\mathcal{U}$*-glutos* $\mathcal{C}$ *the presite* $\mathrm{SG}\mathcal{C} = \mathcal{L}\mathcal{M}\mathcal{C}$ *is, in fact, an* $\mathcal{U}$*-glutos.*

Remark 11.12 Let $\mathcal{C}$ be an $\mathcal{U}$-presite such that $\mathcal{M}\mathcal{C}$ (and, hence, $\mathrm{SG}\mathcal{C}$) is an $\mathcal{U}$-presite. Consider the arrow

$$\widetilde{\mathrm{SG}\mathcal{C}} \longrightarrow \mathrm{SG}\widetilde{\mathcal{C}}, \tag{12}$$

obtained from the universal arrow $Y_{\mathcal{C}}: \mathcal{C} \longrightarrow \widetilde{\mathcal{C}}$ by applying the pseudofunctor $\sim \circ\mathrm{SG}$ to it and using Prop. 11.20 afterwards. The arrow (12) is fully faithful if the pretopology of $\mathcal{C}$ is subcanonical; the inclusion arrow (10) of sect. 10 is the particular case of the arrow (12).

The following addition to Theorem 10.13 states that the set of SG-presites is stable by the reflection $\sim$:

Proposition 11.21 *If* $\mathcal{C}$ *is an SG-presite, then the universal glutos for* $\mathcal{C}$ *is an SG-glutos.*

It is clear from above that any SG-glutos can be obtained as a universal glutos for some $\mathcal{M}$-presite $\mathcal{C}$ and that the corresponding universal arrow $Y_{\mathcal{C}}: \mathcal{C} \longrightarrow \widetilde{\mathcal{C}}$ for an $\mathcal{M}$-$\mathcal{U}$-presite $\mathcal{C}$ can be pulled through the presite $\mathcal{M}\widetilde{\mathcal{C}}$ as

$$\mathcal{C} \xrightarrow{Y'_{\mathcal{C}}} \mathcal{M}\widetilde{\mathcal{C}} \hookrightarrow \mathcal{L}\mathcal{M}\widetilde{\mathcal{C}}, \tag{13}$$

where $Y'_{\mathcal{C}} = \mathcal{M}Y_{\mathcal{C}}$.

Call an $\mathcal{U}$-presite $\mathcal{C}$ **nearly** $\mathcal{U}$**-glutos** if it is naturally equivalent to a presite $\mathcal{M}\widetilde{\mathcal{C}}$ for some $\mathcal{U}$-presite $\mathcal{C}$. Meditating over the decomposition (13) one can conclude that the full 2-subcategory of nearly $\mathcal{U}$-glutoses is reflective in that of all $\mathcal{M}$-$\mathcal{U}$-presites, whereas the arrow $Y'_{\mathcal{C}}$ in (13) is the unit of the corresponding adjunction. Besides, the construction of universal glutos for a nearly $\mathcal{U}$-glutos $\mathcal{C}$ consists simply in adding of all locally clopen arrows to the set of clopen arrows.

The next proposition giving an "internal" description of nearly $\mathcal{U}$-glutoses is a kind of "Giraud theorem" for them.

Proposition 11.22 *Let* $(\mathcal{C}, \tau)$ *be an* $\mathcal{M}$*-presite such that* $\mathcal{C}$ *is an* $\mathcal{U}$*-category and the pretopology* τ *is subcanonical. Then* $(\mathcal{C}, \tau)$ *is a nearly* $\mathcal{U}$*-glutos iff the set* $\mathcal{O}_\tau$ *of clopen arrows of it satisfies condition* $(\mathrm{G4}_{\mathcal{U}} + 5)$ *as well as the following conditions:*
(NG1) *For any object* X *of* $\mathcal{C}$ *the set of clopen subobjects of* X *is* $\mathcal{U}$*-small;*
(NG2) *Any family* $\{U_i \xrightarrow{u_i} X\}_{i\in I}$ *of clopen arrows has a factorization into a covering* $\{U_i \xrightarrow{u'_i} U.\}_{i\in I}$ *followed by a clopen arrow* $U. \xrightarrow{u.} X$ (in particular, unions of arbitrary families of clopen subobjects of X exist (in the lattice of all subobjects of X) and are clopen);
(NG3) *Any epi sink consisting of clopen arrows is a covering of* τ (and, hence, is universal effective epi).

Note that the pretopology of a nearly $\mathcal{U}$-glutos is uniquely recovered from the underlying clopos structure (just as in the case of $\mathcal{U}$-glutoses), so that we will consider nearly $\mathcal{U}$-glutoses either as presites or as closposes, depending on circumstances.

Remark 11.13 The definition of glueing data (see sect. 8) with values in mono's of a category $\mathcal{C}$ can be essentially simplified. Namely, define for any set I the category $\Gamma' I$ as follows. The set of objects of $\Gamma' I$ is the free commutative idempotent monoid $W(I)/R$ over I (i.e. the set of "relations" R consists of two relations: $X^2 = X$ and $XY = YX$). For any objects X and Y of $\Gamma' I$ there exists the only arrow $X \longrightarrow Y$ iff there exists Z such that $YZ = X$. Let n be a natural number. Denote $\Gamma_n I$ the full subcategory of $\Gamma' I$ consisting of all monomials over the variables from I of degree $\leq n$, with the neutral element of the monoid $W(I)/R$ excluded; let $\Gamma_+ I$ be the union of all $\Gamma_n I$ (it will be supposed further that the (discrete) category $\Gamma_1 I$ coincides with the set I).

Now call a functor $U\colon \Gamma_2 I \longrightarrow \mathcal{C}$ with values in mono's of $\mathcal{C}$ an **$\mathcal{M}$-gluon** or **$\mathcal{M}$-glueing data** if there exists its continuation on $\Gamma_3 I$ which respects pullbacks existing in $\Gamma_3 I$ ("cocicle condition"); it then follows that there exists a continuation U_+ of U on the whole $\Gamma_+ I$ respecting pullbacks of $\Gamma_+ I$ and U_+ is unique up to a functor isomorphism.

It is evident enough that one can replace, in the context of $\mathcal{M}$-presites or nearly $\mathcal{U}$-glutoses, open glueing data by "equivalent" $\mathcal{M}$-glueing data.

The following proposition generalizes the realization of sheaves over topological space X as sheaves of sections of corresponding fibre bundles over X.

Proposition 11.23 *Let $(\mathcal{C}, \mathcal{O})$ be a nearly $\mathcal{U}$-glutos and $\mathcal{L}\mathcal{O}$ be the set of all locally clopen arrows of it. Then for any object X of $\mathcal{C}$ the category $\mathcal{L}\mathcal{O}/X$ is naturally equivalent to the SG-topos $\mathrm{Sh}X$ of sheaves over the complete Heyting algebra $\mathcal{O}(X)$ of clopen subobjects of X.*

Indications to the proof. The corresponding natural equivalence $J\colon \mathrm{Sh}X \xrightarrow{\approx} \mathcal{L}\mathcal{O}/X$ can be constructed as follows. Let $s\colon \mathcal{O}(X) \longrightarrow \mathcal{O}/X$ be some natural equivalence selecting for any clopen subobject u of X a clopen arrow $su\colon U \longrightarrow X$ representing this subobject. Given a sheaf $F\colon \mathcal{O}(X) \longrightarrow \mathbf{Set}_{\mathcal{U}}$, we want to construct a locally clopen arrow $JF\colon E \longrightarrow X$ such that its sheaf of sections $(u \in \mathcal{O}(X) \mapsto [su, JF])$ is isomorphic to F. As a first approximation to JF one can take the coproduct (in $\mathcal{L}\mathcal{O}/X$):

$$p = \coprod_{u \in \mathcal{O}(X)} F(u) \otimes su,$$

where $S \otimes Y$ means the coproduct of the family $\{Y\}_{i \in S}$ (copower of Y). Unfortunately, p has too many sections as compared to F, so that to obtain JF from p one needs to "glue together" any two summands of p along the maximal clopen arrow where they are to coincide.

Now we will go from informal considerations above to the formal constructions. Define first the "index set" I_F as

$$I_F = \coprod_{u \in \mathcal{O}(X)} F(u);$$

define a partial order relation on I_F such that for any pair $\langle u, x\rangle$, $\langle v, y\rangle$ ($u, v \in \mathcal{O}(X)$, $x \in F(u)$, $y \in F(v)$) of elements of I_F one has that $\langle u, x\rangle \leq \langle v, y\rangle$ iff $u \leq v$ and $x = \rho^v_u y$, where, of course, $\rho^v_u\colon F(v) \longrightarrow F(u)$ are the corresponding restriction maps of the sheaf F.

For any pair $i = \langle u, x \rangle$ and $j = \langle v, y \rangle$ of elements of I_F there exists the intersection $i \wedge j = \langle w, z \rangle$, where $w \leq u \wedge v$ is the biggest element of $\mathcal{O}(X)$ such that $\rho^u_w x = \rho^v_w y$ and $z = \rho^u_w x$; note that this property of I_F essentially depends on the fact that F is a sheaf and not simply a presheaf.

There exists the only functor

$$\varphi: \Gamma_+ I_F \longrightarrow I_F$$

such that φ is the identity map on $I_F = \Gamma_1 I_F \subset \Gamma_+ I_F$ and, besides, for any pair i, j of elements of I_F the identity $\varphi(ij) = i \wedge j$ holds. Recall that the category $\Gamma_+ I$ is defined in Remark 11.13 above and that I_F is a category being a partially ordered set.

There is as well an evident forgetful functor $I_F \longrightarrow \mathcal{O}(X)$ ($\langle u, x \rangle \mapsto u$), which produces some functor $N: I_F \longrightarrow \mathcal{LO}/X$, when being composed with the chain of functors

$$\mathcal{O}(X) \xrightarrow{s} \mathcal{O}/X \hookrightarrow \mathcal{LO}/X.$$

Composing now the functor N with the restriction of the functor φ (constructed above) on the subcategory $\Gamma_2 I_F$ of $\Gamma_+ I_F$ one obtains some functor

$$U_F: \Gamma_2 I_F \longrightarrow \mathcal{LO}/X.$$

One can verify easily that the functor U_F is an $\mathcal{M}$-gluon, whereas its colimit $U_{F\cdot}$ in $\mathcal{LO}/X$ can play the role of the locally clopen arrow JF corresponding to the sheaf F.$\blacksquare$

Remark 11.14 In an earlier author's work [18] the term "$\mathcal{U}$-glutos" meant something which is now called "nearly $\mathcal{U}$-glutos", whereas $\mathcal{M}$-$\mathcal{U}$-presites with a subcanonical pretopology were called there "$\mathcal{U}$-preglutoses"; the main result of [18] was, in this terms, that every $\mathcal{U}$-preglutos has a universal completion to an $\mathcal{U}$-glutos, whereas its proof has used generalized "charts and atlases routine" (see the next section). Later on it was observed the presence of SG-toposes "inside" glutoses just via Prop.11.23, and the natural question arose how to generalize both the very notion of glutos (so that *arbitrary* toposes can occur in place of SG-toposes) and the theorem of existence of universal glutoses (charts and atlases method failed to prove Theorem 10.13 due to the reasons explained in Appendix A). Sect. A Sect. A).

12 Charts and Atlases

In this section a way of constructing of universal glutoses (or, rather, of nearly glutoses) by means of charts and atlases is considered, applicable for $\mathcal{M}$-presites with subcanonical pretopology.

Give first some necessary definitions. A continuous functor $J: \mathcal{C} \longrightarrow \mathcal{D}$ between presites will be said **to reflect coverings** if for any family $\{u_i: U_i \longrightarrow X\}_{i \in I}$ of clopen arrows of $\mathcal{C}$ the fact that $\{Ju_i: JU_i \longrightarrow JX\}_{i \in I}$ is a covering in $\mathcal{D}$ implies that $\{u_i: U_i \longrightarrow X\}_{i \in I}$ is a covering in $\mathcal{C}$; if, besides, $\mathcal{C}$ is an $\mathcal{M}$-presite with subcanonical pretopology and the underlying functor of J is faithful then J will be said to **admit atlases**.

An $\mathcal{M}$-presite with subcanonical pretopology will be called a **DG-presite** if it satisfies the factorization condition (NG2) in Prop.11.23 above for arbitrary sinks of clopen arrows

(DG above deciphers as "differential-geometrical", because presites of this kind are typical just for differential geometry).

Let $\mathcal{C}$ be an $\mathcal{M}$-$\mathcal{U}$-presite with subcanonical pretopology and $J\colon \mathcal{C} \longrightarrow \mathcal{D}$ be an arrow admitting atlases, such that $\mathcal{D}$ is a nearly $\mathcal{U}'$-glutos, where the universe $\mathcal{U}'$ is any universe containing $\mathcal{U}$ as a subset. In constructing the universal nearly $\mathcal{U}$-glutos for $\mathcal{C}$ one can use the arrow J considering objects of $\widetilde{\mathcal{C}}$ as objects of $\mathcal{D}$ with additional structure.

In fact, the process of completion of $\mathcal{C}$ to $\widetilde{\mathcal{C}}$ using the arrow J can be performed in two steps: first, one completes $\mathcal{C}$ with objects which are "unions of families of clopen subobjects", arriving to a universal DG-presite, associated with $\mathcal{C}$; the second step is the completion of the DG-presite so obtained with objects, which are colimits of clopen $\mathcal{M}$-gluons. Only the second step will be described below, because it occurs very frequently in practice.

So let us assume that the arrow $J\colon \mathcal{C} \longrightarrow \mathcal{D}$ admitting atlases is given such that $\mathcal{C}$ is a DG-$\mathcal{U}$-presite, whereas $\mathcal{D}$ is a DG-$\mathcal{U}'$-presite, where the universe $\mathcal{U}'$ contains the universe $\mathcal{U}$ as a subset.

Let X be an object of $\mathcal{D}$. An $\mathcal{U}$-small family $\{U_i\}_{i\in I}$ of objects of $\mathcal{C}$ together with a covering $\{JU_i \xrightarrow{u_i} X\}_{i\in I}$ of X will be called an **J-atlas on** X if for every $i, j \in I$ the pullback $JU_i \coprod_X JU_j$ has a representation

$$\begin{tikzcd} JU_{ij} \arrow[r, "Ju'_j"] \arrow[d, "Ju'_i"'] & JU_j \arrow[d, "u_j"] \\ JU_i \arrow[r, "u_i"] & X \end{tikzcd} \tag{14}$$

such that both u'_i and u'_j are clopen arrows of $\mathcal{C}$. Any arrow u_i will be called a **chart** of the corresponding J-atlas.

We will identify further a sink $\{JU_i \xrightarrow{u_i} X\}_{i\in I}$ with a J-atlas, omitting its first component $\{U_i\}_{i\in I}$; we will write as well "atlas" instead of "J-atlas", when this will not lead to confusion.

Remark 12.15 The fact that J admits atlases imply that if clopen arrows $U \xrightarrow{u} V$ and $U' \xrightarrow{u'} V$ are such that both Ju and Ju' represent one and the same clopen subobject of JV then u and u' represent one and the same subobject of V (i.e. there exists an iso i such that $u' = ui$).

In particular, clopen arrows u'_i in the definition of atlases above (see the pullback (14)) are, essentially, unique, determining, thus, some clopen $\mathcal{M}$-glueing data in $\mathcal{C}$ such that X is their "colimit in $\mathcal{D}$".

Given atlases A and A' on X we will say that A is **compatible with** A' if the union sink $A \cup A'$ (whose definition is evident) is an atlas on X as well. One can prove that the relation between atlases just defined is, in fact, an equivalence relation; the equivalence class of an atlas A will be denoted further $[A]$.

Let $A = \{JU_i \xrightarrow{u_i} X\}_{i\in I}$ be an atlas on X and $B = \{JV_k \xrightarrow{v_k} Y\}_{k\in K}$ be an atlas on Y. An arrow $f\colon X \longrightarrow Y$ will be called **A-B-admissible** if for any chart u_i of the atlas

A and for any chart v_k of the atlas B the pullback of v_k along fu_i has a representation

$$\begin{tikzcd} JW_{ik} \arrow[rr, "Jf_{ik}"] \arrow[d, "Jw_{ik}"'] & & JV_k \arrow[d, "v_k"] \\ JU_i \arrow[r, "u_i"] & X \arrow[r, "f"] & Y \end{tikzcd} \tag{15}$$

such that w_{ik} is a clopen arrow of $\mathcal{C}$.

Proposition 12.24 *If an arrow $f: X \longrightarrow Y$ of $\mathcal{D}$ is A-B-admissible for some atlases A and B, then f is A'-B'-admissible for any atlases $A' \in [A]$ and $B' \in [B]$; if, besides, an arrow $g: Y \longrightarrow Z$ is B-C-admissible, then the composition arrow gf is A-C-admissible.*

The latter proposition justifies correctness of the following definitions and constructions. First, call the arrow f above $[A]$-$[B]$-**admissible** if it is A-B-admissible. Now one can define the category $\mathcal{C}_J$ as follows. Objects of $\mathcal{C}_J$ are all pairs $\langle X, [A]\rangle$ consisting of an object X of $\mathcal{D}$ and an equivalence class $[A]$ of atlases on it. Arrows of $\mathcal{C}_J$ are all triples $\langle\langle X, [A]\rangle, f, \langle Y, [B]\rangle\rangle$ such that the arrow $f: X \longrightarrow Y$ is $[A]$-$[B]$-admissible (and we will write simply f instead of the whole triple in situations not leading to confusion).

There are evident functors $J_{\mathcal{C}}: \mathcal{C} \longrightarrow \mathcal{C}_J$ ($X \mapsto \langle JX, [\{Id_X\}]\rangle$) and $J': \mathcal{C}_J \longrightarrow \mathcal{D}$ (forget atlases). There is the natural pretopology on $\mathcal{C}_J$ making both $J_{\mathcal{C}}$ and J' continuous. This pretopology is defined as follows. Declare a monic arrow $f: X \longrightarrow Y$ between objects $\langle X, [A]\rangle$ and $\langle Y, [B]\rangle$ of $\mathcal{C}_J$ J-**clopen** if all arrows f_{ik} in the diagram (15) above are clopen arrows of $\mathcal{C}$ (it follows then from the condition (NG2) that f is a clopen arrow of $\mathcal{D}$). Let τ consists of all sinks S in $\mathcal{C}_J$ such that any arrow of S is J-clopen and $J'S$ is a covering in $\mathcal{D}$.

One can prove that τ is really a pretopology on $\mathcal{C}_J$ and the functors $J_{\mathcal{C}}$ and J' become continuous if one equips the category $\mathcal{C}_J$ with the pretopology τ. The notations $\mathcal{C}_J$, $J_{\mathcal{C}}$ and J' will be reserved as well to denote the corresponding presite and morphisms of presites. Note that the equality $J = J'J_{\mathcal{C}}$ holds.

Now, at last, one can formulate the theorem giving a construction of universal nearly $\mathcal{U}$-glutos by means of charts and atlases.

Theorem 12.25 *Let $\mathcal{C}$ be a* DG-$\mathcal{U}$*-presite, $\mathcal{D}$ be a* DG-$\mathcal{U}'$*-presite for $\mathcal{U} \subset \mathcal{U}'$. Let the arrow $J: \mathcal{C} \longrightarrow \mathcal{D}$ admits atlases. Then the presite $\mathcal{C}_J$ constructed above is a* DG-$\mathcal{U}$*-presite. If, moreover, $\mathcal{D}$ is a nearly $\mathcal{U}'$-glutos then $\mathcal{C}_J$ is a universal nearly $\mathcal{U}$-glutos for $\mathcal{C}$, whereas the arrow $J_{\mathcal{C}}: \mathcal{C} \longrightarrow \mathcal{C}_J$ is a corresponding universal arrow.*

Applying this theorem to standard constructions of differential geometry (manifolds, vector bundles, principal G-bundles, etc.,) one can check that all this constructions are just particular cases of universal (nearly) $\mathcal{U}$-glutos construction. But to check that certain functors of algebraic geometry like Spec above fall as well into this scheme, one needs another tools. The theorem below gives sufficient criteria for an arrow between presites to be universal.

Before formulating this theorem one needs to introduce one more definition. An arrow $F: \mathcal{C} \longrightarrow \mathcal{D}$ will be said to **locally reflect clopens** if for any arrow $u: U \longrightarrow X$ of $\mathcal{C}$ the fact that Fu is clopen implies that u is locally clopen.

Theorem 12.26 *Let $\mathcal{C}$ be an $\mathcal{M}$-presite with a subcanonical pretopology, $\mathcal{D}$ be a nearly $\mathcal{U}$-glutos and $Y\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a continuous functor. Then the following statements are equivalent:*
(a) *Y is a universal arrow for $\mathcal{C}$;*
(b) *Y is fully faithful, reflects coverings, locally reflects clopens and, besides, for every object D of $\mathcal{D}$ there exists an $\mathcal{U}$-small covering $\{u_i\colon YU_i \longrightarrow D\}_{i\in I}$ of D by "objects of $\mathcal{C}$".*

Now the universality of the arrow Spec can be established just with the help of Theorem 12.26. This theorem can be applied as well to obtain necessary and sufficient conditions for a given $\mathcal{U}$-valued functor on the category **Ring** to be representable by a Grothendieck scheme; these conditions can be formulated in terms of Zariski pretopology on the category $\mathbf{Ring}^{op}$. (cf. the existence problem of Grothendieck as formulated in [21]).

A The idea of the proof of Th.10.13

Let a set $\mathcal{U}'$ be a universe such that $\mathcal{U} \subset \mathcal{U}'$ and $\mathcal{C}$ is $\mathcal{U}'$-small (recall that we are living, due to Sect. 2, in "Grothendieck's paradise" restricted from above by the universal class of all sets). Let $\mathrm{Sh}_{\mathcal{U}'}\mathcal{C}$ be the topos of $\mathcal{U}'$-valued sheaves on $\mathcal{C}$, considered as a presite via canonical pretopology τ.

In constructing the universal arrow $Y_{\mathcal{C}}\colon \mathcal{C} \longrightarrow \widetilde{\mathcal{C}}$ the Yoneda functor $Y\colon \mathcal{C} \longrightarrow \mathrm{Sh}_{\mathcal{U}'}\mathcal{C}$ can be used, whereas $\mathcal{U}'$-valued sheaves can be considered as building blocks in the process of construction of $\widetilde{\mathcal{C}}$.

In more details, let T be a (non-elementary) theory whose axioms are axioms of elementary theory of categories together with axioms (PT1)–(PT4) of presites and conditions (G2), (G4$_{\mathcal{U}}$), (G5) and (G6$_{\mathcal{U}}$) (as well as (G5P) in case of weak form of (G5)) imposed on the set of clopen arrows (these conditions are the same as in "Giraud theorem" 10.11, *excepting* the $\mathcal{U}$-smallness condition (G7$_{\mathcal{U}}$)). The presite $\mathrm{Sh}_{\mathcal{U}'}\mathcal{C}$ is easily seen to be a model of the theory T and one can prove that submodels of T form a complete lattice with respect to the inclusion functors. The latter lattice is, essentially, a closure system on the set

$$X = \mathrm{Mor}(\mathrm{Sh}_{\mathcal{U}'}\mathcal{C}) \coprod \tau \ .$$

One can prove that the T-closure of the image $Y\mathcal{C}$ of $\mathcal{C}$ by Yoneda functor in $\mathrm{Sh}_{\mathcal{U}'}\mathcal{C}$ is not only model of T but satisfies the condition (G7$_{\mathcal{U}}$) as well. In other words, it is an $\mathcal{U}$-glutos and one can show further that it is the universal glutos $\widetilde{\mathcal{C}}$.

In proving this it is useful to "translate" axioms of the theory T into the set of *inference rules* on the set X (in the sense of [1]), whereas arrows and coverings in $Y\mathcal{C}$ to consider as *axioms* of the corresponding (infinitary) formal system (denoted further $FS(T)$). Then the T-closure of $Y\mathcal{C}$ in $\mathrm{Sh}_{\mathcal{U}'}\mathcal{C}$ turns out to be, essentially, the set of *theorems* of the formal system $FS(T)$.

It is convenient (as well as more informative) to separate the subtheory T_{sub} of "presites with subcanonical pretopology" in T; considering first the T_{sub}-closure of $Y\mathcal{C}$ one can prove that the full sub-2-category of subcanonical $\mathcal{U}$-presites is reflective in $\mathbf{Psite}_{\mathcal{U}}$.

This reduces the proof of Theorem 10.13 to the particular case of $\mathcal{U}$-presites $\mathcal{C}$ with subcanonical pretopology.

In proving that both T_{sub}-closure $\mathcal{C}_{sub}$ and T-closure $\widetilde{\mathcal{C}}$ of YC in $\mathrm{Sh}_{\mathcal{U}'}\mathcal{C}$ have $\mathcal{U}$-small local sets of topological generators (see condition (LTG$_{\mathcal{U}}$) of sect. 9) the following Lemma, easily deduced from Lemme 3.1 on p.231 of [10], is crucial:

Lemma A.27 *Let $\mathcal{C}$ be an $\mathcal{U}$-presite. Then the Yoneda map $Y\colon \mathcal{C} \longrightarrow \mathrm{Sh}_{\mathcal{U}'}\mathcal{C}$ has the following property: for any objects X and X' of $\mathcal{C}$ and any arrow $f\colon YX \longrightarrow YX'$ there exists a covering $\{V_i \xrightarrow{v_i} X\}_{i\in I}$ in $\mathcal{C}$ such that the set of objects $\{V_i : i \in I\}$ is a subset of the set G_X of topological generators over X and for any $i \in I$ there exists an arrow $v_i'\colon V_i \longrightarrow X'$ such that the identity $f\circ Yv_i = Yv_i'$ holds.*

It is just applications of this lemma in transfinite induction on the length of *proofs* (in formal systems $FS(T_{sub})$ and $FS(T)$) which permits one to prove that both $\mathcal{C}_{sub}$ and $\widetilde{\mathcal{C}}$ are $\mathcal{U}$-presites. Moreover, one can prove that any object, arrow and covering of the T_{sub}-closure $\mathcal{C}_{sub}$ has a *finite* proof, which permits one to describe the presite $\mathcal{C}_{sub}$ explicitly.

Now the universality properties of the corresponding arrows $Y'_{\mathcal{C}}\colon \mathcal{C} \longrightarrow \mathcal{C}_{sub}$ and $Y_{\mathcal{C}}\colon \mathcal{C} \longrightarrow \widetilde{\mathcal{C}}$ follow from that of "sheafified Yoneda functors" if one applies (transfinite) induction on the length of proofs: given a continuous functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ into a subcanonical $\mathcal{U}$-presite, resp. into an $\mathcal{U}$-glutos one has that any element Z (arrow or covering) of $\mathcal{C}_{sub}$, resp. of $\widetilde{\mathcal{C}}$ having a proof P, where a family of axioms $\{A_i\}_{i\in I}$ from $\mathcal{C}$ were used, goes by the functor

$$\mathrm{Sh}_{\mathcal{U}'}(F)\colon \mathrm{Sh}_{\mathcal{U}'}(\mathcal{C}) \longrightarrow \mathrm{Sh}_{\mathcal{U}'}(\mathcal{D})$$

into an element Z' which has "the same" proof in $\mathrm{Sh}_{\mathcal{U}}(\mathcal{D})$ as Z has in $\mathrm{Sh}_{\mathcal{U}}(\mathcal{C})$ with only the family $\{A_i\}_{i\in I}$ of axioms replaced by the family $\{FA_i\}_{i\in I}$. This implies that Z' belongs to the closure of $\mathcal{D}$ (naturally equivalent to $\mathcal{D}$, because $\mathcal{D}$ is a model of T_{sub}, resp. of T), i.e. the restriction of the functor $\mathrm{Sh}_{\mathcal{U}}(F)$ on $\mathcal{C}_{sub}$, resp. on $\widetilde{\mathcal{C}}$ can be pulled through $\mathcal{D}$.

It turns out, that if $\mathcal{C}$ is an SG-presite with subcanonical pretopology, then any theorem of the formal system $FS(T)$ has a proof of a fixed finite length. In this case one can use as well another continuous functors $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ in place of Yoneda functor in constructing of $\widetilde{\mathcal{C}}$ (namely, functors admitting atlases defined in sect. 12 above).

B Precloposes and Generalized Grothendieck Topologies

A category $\mathcal{C}$ together with a class $\mathcal{O} \subset \mathcal{C}_1$ of arrows of $\mathcal{C}$ will be called a *preclopos* iff $\mathcal{O}$ contains all isomorphisms of $\mathcal{C}$, is closed with respect to composition of arrows and, besides, satisfies the following condition:

(qp) For any commutative square

$$\begin{array}{ccc} Z & \xrightarrow{\alpha} & U \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle u} \\ Y & \xrightarrow{f} & X \end{array} \qquad (16)$$

such that $u \in \mathcal{O}$ there exists a commutative diagram

$$
\begin{tikzcd}
Z \arrow[rr, "\alpha"] \arrow[dd, "\beta"'] \arrow[dr, dashed] & & U \arrow[dd, "u"] \\
& V \arrow[ur, dashed] \arrow[dl, dashed, "v"'] & \\
Y \arrow[rr, "f"] & & X
\end{tikzcd}
\tag{17}
$$

with $v \in \mathcal{O}$.

Any clopos is a preclopos, evidently.

A sieve $R \subset X$ on an object X of a preclopos $\mathcal{C}$ will be said to be $\mathcal{O}$-*generated* or an $\mathcal{O}$-*sieve* if there exists a family $\{U_i \xrightarrow{u_i} X\}_{i \in I}$ of arrows of $\mathcal{O}$ such that the following condition is satisfied: an arrow f belongs to R iff there exists $i \in I$ such that f pulls through u_i:

$$
\begin{tikzcd}
& U_i \arrow[dr, dashed, "u_i"] & \\
Y \arrow[ur, dashed] \arrow[rr, "f"] & & X
\end{tikzcd}
\tag{18}
$$

Proposition B.28 *For any $\mathcal{O}$-sieve $R \subset X$ and any arrow $f\colon Y \longrightarrow X$ the sieve f^*R is an $\mathcal{O}$-sieve.*

Proposition B.28 justifies the following definition.

A class $\tau = \bigcup\{\tau X : X \in \mathcal{C}_0\}$ of $\mathcal{O}$-sieves on a preclopos $(\mathcal{C}, \mathcal{O})$ will be called an $\mathcal{O}$ - *Grothendieck topology* on $(\mathcal{C}, \mathcal{O})$ or, simply, an $\mathcal{O}$-*topology* if:

(GT1) For any object $X \in \mathcal{C}_0$ the sieve $X \subset X$ belongs to τ;

(GT2) For any $R \in \tau X$ and any arrow $f\colon Y \longrightarrow X$ the sieve f^*R belongs to τY;

(GT3) Let $R \in \tau X$ and $R' \subset X$ be an $\mathcal{O}$-sieve. If for any $v\colon Y \longrightarrow X$ in R the sieve v^*R' belongs to τY, then R' belongs to τX (local character).

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